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Iterative Techniques for Solving Positive Triangular Fully Fuzzy Linear System

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Abstract

In this paper, we propose two numerical methods, namely: Gauss– Jacobi and Gauss–Seidel iterative schemes to solve the Positive Triangular Fully Fuzzy Linear System. A description of these numerical schemes and their convergence properties have been presented. Numerical example with known exact solution are given to illustrate the efficiency of the proposed methods. Numerical results show clearly that the Gauss–Seidel method provides more efficient results than its counterparts.

Keywords: Fully fuzzy linear system, Iterative schemes, Fuzzy solution.
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1. Introduction

Fuzzy linear system (FLS) with a crisp real coefficient matrix and with a vector of fuzzy triangular numbers on the right-hand side has a wide application in varying subjects including mathematics, physics, economics, statistics, telecommunications, image processing and even social sciences. A general method for solving FLS where coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy vector was first proposed by Friedman et al. [8]. They used the embedding method and replaced the original fuzzy linear system by a crisp linear system and obtained the solution. Dehghan et al. [5, 6] and [10] have used some iterative methods to solve the same system. Here we will discuss the case in which all parameters in a fuzzy linear system are fuzzy numbers. This system is called Fully Fuzzy Linear System (FFLS). In[7] Dehghan et al. introduced several methods for solving FFLS that are similar to the classical methods based on linear algebra, such as the LU decomposition and crammer's rule in order to determine the approximate solution of the system. The Adomian decomposition method was also implemented in order to solve the positive fuzzy vector solution of FFLS [2, 4, 10]. Dehghan et al. [4, 5] investigated iterative schemes like Gauss−Seidel, Jacobi and

Jacobi over− relaxation (JOR) for solving FFLS. Malkawi et al.[12] have proposed new matrix methods for solving positive FFLS. In [11] Kumar et al. introduced a new computational method to solve FFLS by relying on the computation of row reduced echelon form. Buckley and Qu[2] studied 1−cut of an FFLS and proposed different solutions for it. Based on their work, Muzzioli and Reynaerts[13] investigated an FFLS in the dual form :

$$\tilde{A}_1\tilde{x} + \tilde{b}_1 = \tilde{A}_2\tilde{x} + \tilde{b}_2$$

Their approach leads to finding a relation between interval linear system and fuzzy linear system. Also, they proposed to solve $2^{n(n+1)}$ crisp linear systems. Recently, Allahviranloo et al. [1] proposed a new method to obtain symmetric solutions of FFLS based on the 1−cut expansion which lead to compute the solution of $2n$ linear equations. Ineirat [9] investigated the numerical handling of the fuzzy linear system of equations (FSLEs) and fully fuzzy linear system (FFLS).

In this paper, we present two iterative techniques, namely: Gauss– Jacobi and Gauss–Seidel schemes for FFLS using MATLAB software.

The outline of the paper is as follows: in section 2 we present some basic definitions of fully fuzzy linear system. In section 3, two iterative schemes, namely the Gauss– Jacobi and Gauss–Seidel schemes are presented. section 4 contains one numerical example intending to show the efficiency of proposed iterative methods. Conclusion is drawn in Section 5.

2. Preliminaries

In this section, we review some basic definitions of fuzzy theory [5, 12].

Definition 1. Let X denote a universal set. Then a fuzzy subset $\breve{A}$ of X is defined by its membership function

$$\mu_{\breve{A}}: X \to [0,1]$$

which assigns a real number $\mu_{\breve{A}}(x)$ in the interval [0, 1], to each element $x \in X$ where the value of $\mu_{\breve{A}}(x)$ at x shows the grade of membership of $x \in \breve{A}$.

Definition 2. A fuzzy set $\breve{A}$ in $X = R^n$ is said to be a convex fuzzy set if and only if its α −level sets are convex.

Definition 3. A fuzzy set $\breve{A}$ in $X = R^n$is said to be normal if there exist $x \in X$ such that $\mu_{\breve{A}}(x)$= 1.

Definition 4. A fuzzy number is a convex normalized fuzzy set of the real line R^1 where membership function is piecewise continuous.

Definition 5. A fuzzy number M is called positive (negative) denoted by $M > 0$ ($M < 0$) if its membership function $\mu_{\breve{M}}(x)$ satisfies

$$\mu_{\breve{M}}(x) = 0, \forall x < 0 \ (x > 0).$$

Definition 6. A fuzzy number is a fuzzy set $v: R \to [0,1]$ which satisfies:

- v is upper semi continuous.
- $v(x) = 0$ outside some interval $[a, d]$.
- There are real numbers $b, c :\ a \leq b \leq c \leq d$ for which
  a. $v(x)$ is monotonic increasing on $[a, b]$.
  b. $v(x)$ is monotonic decreasing on $[c, d\,]$.
  c. $v(x) = 1,\ b \leq x \leq c$.

$$\text{i.e. } v(x) = \begin{cases} 0, & x \leq a \\ f(x), & a \leq x \leq b \\ 1, & b \leq x \leq c \\ g(x), & c \leq x \leq d \\ 0, & x \geq d \end{cases}$$

where f is an increasing function and is called the left side, while g is a decreasing function and is called the right side.

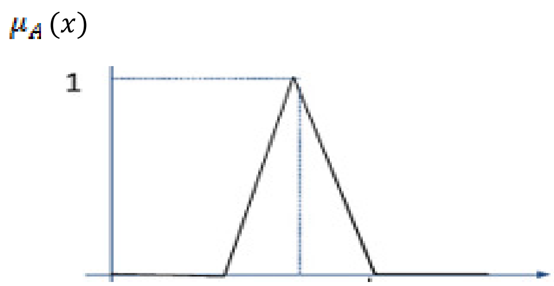
Definition 7. Triangular Fuzzy Number

A triangular fuzzy number (TFN) as illustrated in Figure (1.1) is a special type and the most common of fuzzy number and its membership function $\mu_A(x)$ is given by:

$$\mu_A(x) = \begin{cases} 0 , & x \le a, \\ \frac{x-a}{b-a} , & a \le x \le b, \\ \frac{c-x}{c-b} , & b \le x \le c, \\ 0 , & c \le x . \end{cases}$$

and we denoted it by $\mu_A(x) = (a, b, c)$.

Figure 1: Triangular fuzzy number



Definition.8 For two fuzzy numbers $\tilde{m} = (m, \alpha, \beta)$ and $\tilde{n} = (n, \gamma, \delta)$, we define the following arithmetic operations:

Addition:

$$\tilde{m} \oplus \tilde{n} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta).$$

subtraction:

$$\tilde{m} \ominus \tilde{n} = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m, \alpha, \beta) \oplus (-n, \delta, \gamma) = (m - n, \alpha + \delta, \beta + \gamma).$$

Multiplication of two fuzzy numbers : If $\tilde{m} > 0$ and $\tilde{n} > 0$, then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) = (mn, m\gamma + n\alpha, m\delta + n\beta).$$

Symmetric image $-(\tilde{m}) = (-\beta, -\alpha, -m)$

Definition 9. Two triangular fuzzy numbers $\tilde{m} = (m, \gamma, \beta)$ and $\tilde{n} = (n, \alpha, \delta)$ are said to be equal, if and only if $m = n, \gamma = \alpha$ and $\beta = \delta$.

Definition 10. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number.

Definition 11. A square fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})$ is an upper triangular fuzzy matrix, if $\tilde{a}_{ij} = \tilde{0} = (0,0,0), \forall i > j$. The transpose of an upper triangular fuzzy matrix, is lower triangular; that is $\tilde{a}_{ij} = \tilde{0} = (0,0,0), \forall i < j$.

Definition 12. The $n \times n$ linear system of equations:

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1 \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2 \\ \vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_n) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_n) \oplus \dots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n \end{cases} \quad (1)$$

where the coefficient matrix $\widetilde{A}$, $\tilde{x}$, $\tilde{b} = (\tilde{a}_{ij}) \in \mathbb{I}, 1 \le i,j \le n$ is called a Fully Interval Linear System (FILS).
the matrix form of system(1) is

$$\widetilde{A} \otimes \tilde{x} = \tilde{b} \tag{2}$$

where $\tilde{A} = ([\tilde{a}_{ij}])_{n\times n}$ is an interval number matrix and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T$ are two interval vectors.

3. Numerical Methods

In this section, two iterative methods namely: Gauss−Jacobi, and Gauss−Seidel methods are introduced to solve the (FFLS).

Our goal is to obtain a positive solution of a fully fuzzy linear system (1) where $\tilde{A} = (A, M, N) > 0, \tilde{b} = (b, g, h) > 0$ and $\tilde{x} = (x, y, z) > 0$.

Thus we have :

$$(A, M, N) \otimes (x, y, z) = (b, g, h). \tag{3}$$

Using the approximation formula for the extended multiplication of two fuzzy numbers, then equation (3) may be written as:

$$(Ax, Ay + Mx, Az + Nx) = (b, g, h)$$

This is equivalent to

$$\begin{aligned} Ax &= b, \\ Ay + Mx &= g, \\ Az + Nx &= h. \end{aligned} \tag{4}$$

Theorem 1. [6]: Let $\tilde{A} = (A, M, N)$ and $\tilde{b} = (b, g, h)$ be a nonnegative fuzzy matrix and a nonnegative fuzzy vector, and let A be the product of a permutation matrix by a diagonal matrix with positive diagonal entries. Also, let $h \ge MA^{-1}b$, $g \ge NA^{-1}b$ and $(MA^{-1} + I)b \ge h$. Then the system $\widetilde{A}\tilde{x} = \tilde{b}$ has a positive fuzzy solution.

3.1. Gauss - Jacobi Method. Consider the FFLS

$$\widetilde{A}\tilde{x} = \tilde{b} \tag{5}$$

In virtue of (4), then (5) yields

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ (a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n) + (m_{i1}x_1 + m_{i2}x_2 + \cdots + m_{in}x_n) = g_i, \\ (a_{i1}z_1 + a_{i2}z_2 + \cdots + a_{in}z_n) + (n_{i1}x_1 + n_{i2}x_2 + \cdots + n_{in}\, x_n) = h_i \end{cases} \tag{6}$$

where $1 \le i \le n$.

From system (6) we obtain

$$\begin{cases} a_{ii} x_i = b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j \\ a_{ii} y_i = g_i - \left(\sum_{j=1, j \neq i}^{n} a_{ij} y_j + \sum_{j=1}^{n} m_{ij} x_j \right), \quad 1 \leq i \leq n, a_{ii} \neq 0 \\ a_{ii} z_i = h_i - \left(\sum_{j=1, j \neq i}^{n} a_{ij} z_j + \sum_{j=1}^{n} n_{ij} x_j \right) \end{cases} \tag{7}$$

Hence

$$\begin{cases} x_i = \frac{1}{a_{ii}} (b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j) \\ y_i = \frac{1}{a_{ii}} (g_i - \left(\sum_{j=1, j \neq i}^{n} a_{ij} y_j + \sum_{j=1}^{n} m_{ij} x_j \right)), \quad 1 \leq i \leq n \\ z_i = \frac{1}{a_{ii}} (h_i - \left(\sum_{j=1, j \neq i}^{n} a_{ij} z_j + \sum_{j=1}^{n} n_{ij} x_j \right)) \end{cases} \tag{8}$$

This can easily be written as

$$\begin{cases} x_i = -\frac{1}{a_{ii}} \sum_{j=1, j \neq i}^{n} a_{ij} x_j + \frac{b_i}{a_{ii}} \\ y_i = -\frac{1}{a_{ii}} \left(\sum_{j=1, j \neq i}^{n} a_{ij} y_j + \sum_{j=1}^{n} m_{ij} x_j \right) + \frac{g_i}{a_{ii}}, \quad 1 \leq i \leq n \\ z_i = -\frac{1}{a_{ii}} \left(\sum_{j=1, j \neq i}^{n} a_{ij} z_j + \sum_{j=1}^{n} n_{ij} x_j \right) + \frac{h_i}{a_{ii_i}} \end{cases} \tag{9}$$

Equations (97) can be expressed in the matrix form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \beta \tag{10}$$

where M is called the iteration matrix and β is a vector.

To solve system (10), we start with an initial approximation $X^{(0)}$ to the solution X and generates a sequence $\{X^{(K)}\}_{K=0}^{\infty}$that converges to X. thus we obtain

$$\begin{cases} x_i^{(k+1)} = \dfrac{1}{a_{ii}}\left(b_i - \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij}\, x_j^{(k)}\right) & (11) \\ y_i^{(k+1)} = \dfrac{1}{a_{ii}}\left(g_i - \left(\displaystyle\sum_{j=1, j\neq i}^{n} a_{ij}\, y_j^{(k)} + \sum_{j=1}^{n} m_{ij}\, x_j^{(k)}\right)\right), 1 \leq i \leq n & (12) \\ z_i^{(k+1)} = \dfrac{1}{a_{ii}}\left(h_i - \left(\displaystyle\sum_{j=1, j\neq i}^{n} a_{ij}\, z_j^{(k)} + \sum_{j=1}^{n} n_{ij}\, x_j^{(k)}\right)\right) & (13) \end{cases}$$

In matrix form we obtain

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = M \begin{bmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{bmatrix} + \beta, \quad (k \geq 0) \tag{14}$$

Equation (11) can be written into the matrix form as:

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \cdots \\ \cdots \\ x_n^{(k+1)} \end{bmatrix} = - \begin{bmatrix} \frac{1}{a_{11}} & & & & \\ & \frac{1}{a_{22}} & & & \\ & & \frac{1}{a_{33}} & & \\ & & & \cdots & \\ & & & & \frac{1}{a_{nn}} \end{bmatrix} \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{32} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & 0 & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \cdots \\ \cdots \\ x_n^{(k)} \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_n \end{bmatrix} \right\}$$

or

$$x^{(k+1)} = -D_A^{-1}(L_A + U_A)x^{(k)} + D_A^{-1}b$$

likewise, equation (12) and (13) can be expressed into matrix form respectively as

$$y^{(k+1)} = -D_A^{-1}(L_A + U_A)y^{(k)} + D_A^{-1}g$$
$$z^{(k+1)} = -D_A^{-1}(L_A + U_A)z^{(k)} + D_A^{-1}h$$

3.1.1. Convergence of the Gauss – Jacobi Method

For the convergence of the Gauss-Jacobi iterative scheme we have the following theorems :

Theorem 2. [3] The Gauss -Jacobi iterative method for solving fully fuzzy linear system of equations $\widetilde{A} \otimes \tilde{x} = \tilde{b}$ converges if and only if the classical Gauss- Jacobi iterative method converges for solving the crisp linear system of equations $Ax = b$ derived from the corresponding fully fuzzy linear system of equations.

Theorem 3 [3] If the matrix A in the crisp linear system of equations $Ax = b$ is strictly diagonally dominant i.e., $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, i = 1,2,3,\ldots,n$ then the iterations obtained in the classical Gauss- Jacobi iterative method converges for any initial approximation $X^{(0)}$.

3.2. Gauss- Seidel Method

Another well-known iterative method for solving the FFLS is the Gauss–Seidel method. system (6) can be written as:

$$\begin{cases} \sum_{j\le i} a_{ij}x_j = b_i - \sum_{j>i}^{n} a_{ij}x_j, \\ \sum_{j\le i} a_{ij}y_i = g_i - \left(\sum_{j>i}^{n} a_{ij}y_j + \sum_{j=1}^{n} m_{ij}x_j\right), \qquad 1\le i\le n, \\ \sum_{j\le i} a_{ij}z_i = h_i - \left(\sum_{j>i}^{n} a_{ij}z_j + \sum_{j=1}^{n} n_{ij}x_j\right). \end{cases}$$

After choosing an initial approximation $X^{(0)}$, the Gauss−Seidel method generates a sequence $\{X^{(K)}\}_{K=0}^{\infty}$ given by

$$\begin{cases} x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j<i}^{n} a_{ij}x_j^{(k+1)} - \sum_{j>i}^{n} a_{ij}x_j^{(k)}\right], \\ y_i^{(k+1)} = \frac{1}{a_{ii}}\left[g_i - \sum_{j<i}^{n} a_{ij}y_j^{(k+1)} - \sum_{j>i}^{n} a_{ij}y_j^{(k)} - \sum_{j=1}^{n} m_{ij}x_j^{(k)}\right], 1\le i\le n, k\ge 0 \\ z_i^{(k+1)} = \frac{1}{a_{ii}}\left[h_i - \sum_{j<i}^{n} a_{ij}z_j^{(k+1)} - \sum_{j>i}^{n} a_{ij}z_j^{(k)} - \sum_{j=1}^{n} n_{ij}x_j^{(k)}\right] \end{cases} \qquad (15)$$

In matrix form, the system (15) gives

$$\begin{cases} (D_A + L_A)x^{(k+1)} = b - U_A x^{(k)} \\ (D_A + L_A)y^{(k+1)} = g - U_A y^{(k)} - Mx^{(k)} \\ (D_A + L_A)z^{(k+1)} = h - U_A z^{(k)} - Mx^{(k)} \end{cases}$$

where $D_A, L_A\, and\, U_A$ are diagonal, lower triangular and upper triangular matrices respectively. Therefore the Gauss-Seidel iterative method for solving FFLS yields

$$\begin{cases} x^{(k+1)} = -(D_A + L_A)^{-1}U_A x^{(k)} + (D_A + L_A)^{-1}b \\ y^{(k+1)} = -(D_A + L_A)^{-1}U_A y^{(k)} + (D_A + L_A)^{-1}Mx^{(k)} + (D_A + L_A)^{-1}g \\ z^{(k+1)} = -(D_A + L_A)^{-1}U_A z^{(k)} + (D_A + L_A)^{-1}Nx^{(k)} + (D_A + L_A)^{-1}h \end{cases} \qquad (16)$$

3.2.1. Convergence of the Gauss- Seidel Method

Theorem 4. [3] If A is strictly diagonally dominant matrix, then the Gauss-Seidel method always converges. The Gauss-Seidel method will generally converge if the Jacobi method converges. Moreover, it converges at a faster speed.

4. Illustrative Numerical Examples and Results

To test the efficiency and accuracy of the proposed iterative techniques discussed in section3, we will use Matlab software to solve the following fully fuzzy linear system with known exact solution.

Example1: Consider the following FFLS:

$$(5,1,1) \otimes (x_1, y_1, z_1) \oplus (6,1,2) \otimes (x_2, y_2, z_2) = (50,10,17)$$
(4.1)
$$(7,1,0) \otimes (x_1, y_1, z_1) \oplus (4,0,1) \otimes (x_2, y_2, z_2) = (48,5,7)$$

From the above system we have

$$A = \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and}$$
$$b = \begin{bmatrix} 50 \\ 48 \end{bmatrix} \quad g = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \quad h = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$$

To solve the FFLS using Gauss−Jacobi method we obtain

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 48 \end{bmatrix} \tag{4.2}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \tag{4.3}$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix} \tag{4.4}$$

Equation (4.2) gives the system

$$5x_1 + 6x_2 = 50$$
$$7x_1 + 4x_2 = 48$$

Since$|5| \ngtr |6|$ and $|4| \ngtr |7|$, then the above system of equations is not diagonally dominant. So, writing the above system in the diagonally dominant form as:

$$7x_1 + 4x_2 = 48$$
(4.5)
$$5x_1 + 6x_2 = 50$$

The above system can now be solves for x_1 and x_2 respectively as:

$$x_1 = \frac{1}{7}(48 - 4x_2)$$
(4.6)
$$x_2 = \frac{1}{6}(50 - 5x_1)$$

This system has the exact solution (4,5), iterative method generates a sequence $\{X^{(K)}\}$ as

$$x_1^{(k+1)} = \frac{1}{7}\left(48 - 4x_2^{(k)}\right)$$
$$,k = 0,1,2, \dots$$
$$x_2^{(k+1)} = \frac{1}{6}\left(50 - 5x_1^{(k)}\right)$$

Starting with initial the approximation vector $x^{(0)} = (0,0)$, we get

$$x_1{}^{(1)} = \frac{1}{7}\left(48 - 4x_2{}^{(0)}\right) = \frac{48}{7} = 6.8571$$
$$x_2^{(1)} = \frac{1}{6}\left(50 - 5x_1{}^{(0)}\right) = \frac{50}{6} = 8.3333$$
i.e. $x^{(1)} = (6.8571,\ 8.3333)$

Continuing the process we obtain the results depicted in table (4.1)

Table (4.1): the sequence $x^{(k)}, k = 0,1,2, ...$ generated by the Gauss-Jacobi method, with $\varepsilon = 1 \times 10^{-2}$.

k	$x = (x_1, x_2)$	**Error** $\|x - x_0\|$
1	$x^{(1)} = (6.8571,8.3333)$	10.7919
2	$x^{(2)} = (2.0952, 2.6190)$	7.4383
3	$x^{(3)} = (5.3605,6.5873)$	5.1390
4	$x^{(4)} = (3.0930,3.8662)$	3.5421
5	$x^{(5)} = (4.6479,5.7559)$	2.4471
6	$x^{(6)} = (3.5681,4.4601)$	1.6867
7	$x^{(7)} = (4.3085,5.3599)$	1.1653
8	$x^{(8)} = (3.7943,4.7429)$	0.8032
9	$x^{(9)} = (4.1469,5.1714)$	0.5549
10	$x^{(10)} = (3.9021,4.8776)$	0.3825
11	$x^{(11)} = (4.0700,5.0816)$	0.2642
12	$x^{(12)} = (3.9534,4.9417)$	0.1821
13	$x^{(13)} = (4.0333,5.0389)$	0.1258
14	$x^{(14)} = (3.9778,4.9722)$	0.0867
15	$x^{(15)} = (4.0159, 5.0185)$	0.0599
16	$x^{(16)} = (3.9894,4.9868)$	0.0413
17	$x^{(17)} = (4.0076,5.0088)$	0.0285
18	$x^{(18)} = (3.9950,4.9937)$	0.0197
19	$x^{(19)} = (4.0036, 5.0042)$	0.0136
20	$x^{(20)} = (3.9976, 4.9970)$	0.0094

One sees clearly that the sequence $\{x^{(k)}\}$ will converge to the exact solution $x = (x_1, x_2) = (4.00,5.00)$

Now, putting the value of (x_1, x_2) into the equations (4.3) and (4.4) we obtain

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

i.e. $\begin{aligned} 5y_1 + 6y_2 &= 1 \\ 7y_1 + 4y_2 &= 1 \end{aligned}$ and $\begin{aligned} 5z_1 + 6z_2 &= 3 \\ 7z_1 + 4z_2 &= 2 \end{aligned}$

The above systems are not in the diagonally dominant form. Converting them to the diagonally dominant form :

$7y_1 + 4y_2 = 1$
(4.7)

$5y_1 + 6y_2 = 1$
Following similar procedure as before, we obtain

$$y_1 = \frac{1}{7}(1 - 4y_2)$$
$$y_2 = \frac{1}{6}(1 - 5y_1)$$

Starting with the initial approximation $y^{(0)} = (0,0)$, the and Gauss- Jacobi method generates a sequence $\{ y^{(k)}\}$ shown in table (4.2).

Table (4.2):the sequence $y^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss- Jacobi method, with $\varepsilon = 1 \times 10^{-2}$.

K	$y = (y_1, y_2)$	**Error** $\lvert y - y_0 \rvert$
1	$y^{(1)} = (0.143200, 0.167570)$	0.2204
2	$y^{(2)} = (0.047448, 0.048233)$	0.1530
3	$y^{(3)} = (0.115640, 0.128030)$	0.1050
4	$y^{(4)} = (0.070042, 0.071201)$	0.0729
5	$y^{(5)} = (0.102510, 0.109200)$	0.0500
6	$y^{(6)} = (0.080801, 0.082138)$	0.0347
7	$y^{(7)} = (0.096264, 0.100230)$	0.0238
8	$y^{(8)} = (0.085924, 0.087346)$	0.0165
9	$y^{(9)} = (0.093288, 0.095962)$	0.0113
10	$y^{(10)} = (0.088364, 0.089826)$	0.0079

One sees clearly that the sequence $\{ y^{(k)}\}$will converge to the exact solution
$y = (y_1, y_2) = (0.0909, 0.0909)$

Similarly solving

$5z_1 + 6z_2 = 3$
(4.8)
$7z_1 + 4z_2 = 2$

Starting with the initial approximation $z^{(0)} = (0,0)$, the and Gauss- Jacobi method generates a sequence $\{ z^{(k)}\}$ shown in table (4.3).

Table (4.3):the sequence $z^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss- Jacobi method, with $\varepsilon = 1 \times 10^{-2}$.

K	$z = (z_1, z_2)$	**Error** $\lvert z - z_0 \rvert$
1	$z^{(1)} = (0.28614, 0.5014)$	0.5773
2	$z^{(2)} = (-0.00037113, 0.26295)$	0.3728
3	$z^{(3)} = (0.13589, 0.50171)$	0.2749
4	$z^{(4)} = (-0.00054786, 0.38816)$	0.1775
5	$z^{(5)} = (0.064337, 0.50186)$	0.1309
6	$z^{(6)} = (-0.00063202, 0.44778)$	0.0845
7	$z^{(7)} = (0.030266, 0.50193)$	0.0623
8	$z^{(8)} = (-0.0006721, 0.47618)$	0.0403
9	$z^{(9)} = (0.014041, 0.50196)$	0.0297
10	$z^{(10)} = (-0.00069118, 0.4897)$	0.0192
11	$z^{(11)} = (0.0063151, 0.50197)$	0.0141
12	$z^{(12)} = (-0.00070027, 0.49614)$	0.0091

The sequence $\{z^{(k)}\}$will converge to the exact solution

$$z = (z_1, z_1) = (0,0.5)$$

Hence the solution of given fully fuzzy linear system of equations is as follows:

$$\tilde{x} = \begin{bmatrix} (4,0.0909,0) \\ (5,0.0909,0.5) \end{bmatrix}$$

Which is the required solution of the given fully fuzzy linear system of equations.

5. Gauss Seidel Method

Now, we will attempt to solve example (4.1) using the Gauss− Seidel method i.e.

$$x_1 = \frac{1}{7}(48 - 4x_2)$$
$$x_2 = \frac{1}{6}(50 - 5x_1)$$

The Gauss-Seidel iterative formula for this system can be written as:

$$x_1{}^{(k+1)} = \frac{1}{7}\left(48 - 4x_2{}^{(k)}\right)$$
$$,k = 0,1,2, \ldots$$
$$x_2{}^{(k+1)} = \frac{1}{6}\left(50 - 5x_1{}^{(k+1)}\right)$$

Taking the $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (0,0)$ we get

$$x_1{}^{(1)} = \frac{1}{7}\left(48 - 4x_2{}^{(0)}\right) = \frac{48}{7} = 6.8571$$
$$x_2^{(1)} = \frac{1}{6}\left(50 - 5x_1{}^{(1)}\right) = \frac{15.7145}{6} = 2.6191$$
$$\text{i.e.} \quad x^{(1)} = (6.8571,\ 2.6191)$$

Continuing the process we obtain the results depicted in table (4.4)

Table (4.4):the sequence $x^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss-Seidel method, with $\varepsilon = 1 \times 10^{-2}$.

k	$x = (x_1, x_2)$	Error $\lvert x - x_0 \rvert$
1	$x^{(1)} = (6.8571,2.6190)$	7.3403
2	$x^{(2)} = (5.3605,3.8662)$	1.9481
3	$x^{(3)} = (4.6479\ ,4.4601)$	0.9277
4	$x^{(4)} = (4.3085,4.7429)$	0.4418
5	$x^{(5)} = (4.1469,4.8776)$	0.2104
6	$x^{(6)} = (4.0700,4.9417)$	0.1002
7	$x^{(7)} = (4.0333,4.9722)$	0.0477
8	$x^{(8)} = (3.7943,4.7429)$	0.0227
9	$x^{(9)} = (4.0159,4.9868)$	0.0108
10	$x^{(10)} = (4.0036,4.9970)$	0.0052

One sees clearly that the sequence $\{x^{(k)}\}$ generated by the Gauss-Seidel method will converges to the exact solution $x = (x_1, x_2) = (4,5)$

Starting with the initial approximation $y^{(0)} = (0,0)$, the and Gauss- Seidel method generates a sequence $\{ y^{(k)}\}$ shown in table (4.5).

Table (4.5):the sequence $y^{(k)}, k = 0,1,2, \dots$ generated by the Gauss-Seidel method, with $\varepsilon = 1 \times 10^{-2}$

k	$y = (y_1, y_2)$	Error $\lvert y - y_0 \rvert$
1	$y^{(1)} = (0.142340 ,0.047947)$	0.1502
2	$y^{(2)} = (0.114940, 0.070779)$	0.0357
3	$y^{(3)} = (0.101900, 0.081652)$	0.1070
4	$y^{(4)} = (0.095685, 0.086829)$	0.0081

Hence the value of y up to two decimal places can be written as
$y = (y_1, y_2) = (0.09,0.09)$

Starting with the initial approximation $z^{(0)} = (0,0)$, the and Gauss- Seidel method generates a sequence $\{ z^{(k)}\}$ shown in table (4.6).

Table (4.6):the sequence $z^{(k)}, k = 0,1,2, \dots$ generated by the Gauss-Seidel method, with $\varepsilon = 1 \times 10^{-2}$.

k	$z = (z_1, z_2)$	Error $\lvert z - z_0 \rvert$
1	$z^{(1)} = (0.28614,0.2619500)$	0.53879
2	$z^{(2)} = (0.13646,0.3866800)$	0.1948
3	$z^{(3)} = (0.06518,0.4460800)$	0.0928
4	$z^{(4)} = (0.031238,0.474370)$	0.0442
5	$z^{(5)} = (0.015075,0.487840)$	0.0210
6	$z^{(6)} = (0.0073784,0.49425)$	0.0100
7	$z^{(7)} = (0.0037134,0.49731)$	0.0048

Hence from the above results, we find that the value of z up to two decimal points is found to be:
$z = (z_1, z_2) = (0,0.5)$

Hence the solution of the given fully fuzzy linear system of equation up to two decimal places is found to be

$$\breve{x} = \begin{bmatrix} (4,0.09,0) \\ (5,0.09,0.5) \end{bmatrix}$$

Example 2: Consider the following FFLS:
Consider the following FFLS:

$$(9,0.01,0) \otimes (x_1, y_1, z_1) \oplus (1,0,0.01) \otimes (x_2, y_2, z_2) \oplus (0,0,0.02) \otimes (x_3, y_3, z_3) = (55,7.2,42.07)$$
$$(1,0,0.04) \otimes (x_1, y_1, z_1) \oplus (4,3,0.6) \otimes (x_2, y_2, z_2) \oplus (1,0,0.9) \otimes (x_3, y_3, z_3) = (15,8.3,27.09)$$
$$(0,0,0.1) \otimes (x_1, y_1, z_1) \oplus (2,0,0.3) \otimes (x_2, y_2, z_2) \oplus (4,2,1) \otimes (x_3, y_3, z_3) = (16,11.4,19.4)$$

From the above system we have

$$A=\begin{bmatrix}9&1&0\\1&4&1\\0&2&4\end{bmatrix}\qquad M=\begin{bmatrix}0.01&0&0\\0&3&0\\0&0&2\end{bmatrix}\qquad N=\begin{bmatrix}0&0.1&0.02\\0.4&0.6&0.9\\0.1&0.3&1\end{bmatrix}$$ and

$$b=\begin{bmatrix}55\\15\\16\end{bmatrix}\qquad g=\begin{bmatrix}7.2\\8.3\\11.4\end{bmatrix}\qquad h=\begin{bmatrix}42.07\\27.09\\19.4\end{bmatrix}$$

To solve the FFLS using Gauss−Jacobi method we obtain

$$\begin{bmatrix}9&1&0\\1&4&1\\0&2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}55\\15\\16\end{bmatrix}\tag{18}$$

$$\begin{bmatrix}0.01&0&0\\0&3&0\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}9&1&0\\1&4&1\\0&2&4\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}7.2\\8.3\\11.4\end{bmatrix}\tag{19}$$

$$\begin{bmatrix}0&0.1&0.02\\0.4&0.6&0.9\\0.1&0.3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}9&1&0\\1&4&1\\0&2&4\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}42.07\\27.09\\19.4\end{bmatrix}\tag{20}$$

Equation (18) gives the system

$$\begin{aligned}&9x_1+x_2=55\\&x_1+4x_2+x_3=15\\&2x_2+4x_3=16\end{aligned}\tag{21}$$

System (21) can now be solves for x_1, x_2 and x_3 respectively as:

$$\begin{aligned}x_1&=\frac{1}{9}(55-x_2)\\x_2&=\frac{1}{4}(15-x_1-x_3)\\x_3&=\frac{1}{4}(16-2x_2)\end{aligned}\tag{22}$$

This system has the exact solution (5.9508,1.4426,3.2787), the gauss- Jacobi generates a sequence $\{X^{(K)}\}$ as

$$\begin{aligned}x_1^{(k+1)}&=\frac{1}{9}\left(55-x_2^{(k)}\right)\\x_2^{(k+1)}&=\frac{1}{4}\left(15-x_1^{(k)}-x_3^{(k)}\right),\qquad k=0,1,2,\dots\\x_3^{(k+1)}&=\frac{1}{4}\left(16-2x_2^{(k)}\right)\end{aligned}$$

Starting with initial the approximation vector $x^{(0)}=(0,0,0)$, we get

$$\begin{aligned}x_1^{(1)}&=\frac{1}{9}\left(55-x_2^{(0)}\right)=\frac{55}{9}=6.1111\\x_2^{(1)}&=\frac{1}{4}\left(15-x_1^{(0)}-x_3^{(0)}\right)=\frac{15}{4}=3.75\\x_3^{(1)}&=\frac{1}{4}\left(16-2x_2^{(0)}\right)=\frac{16}{4}=4\end{aligned}$$

i.e. $x^{(1)} = (6.1111,\ 3.75,4)$

Continuing the process we obtain the results depicted in table (1). One sees

Table 1: the sequence $x^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss-Jacobi method, with $\varepsilon = 1 \times 10^{-2}$.

k	$x = (x_1, x_2, x_3)$	Error: $\lvert x - x_0 \rvert$
1	$x^{(1)} = (6.1111 \quad 3.75 \quad 4)$	8.2102
2	$x^{(2)} = (5.6944 \quad 1.2222 \quad 2.125)$	3.1747
3	$x^{(3)} = (5.9753 \quad 1.7951 \quad 3.3889)$	1.4158
4	$x^{(4)} = (5.9117 \quad 1.409 \quad 3.1024)$	0.4850
5	$x^{(5)} = (5.9546 \quad 1.4965 \quad 3.2955)$	0.2163
6	$x^{(6)} = (5.9448 \quad 1.4375 \quad 3.2518)$	0.0741
7	$x^{(7)} = (5.9514 \quad 1.4509 \quad 3.2813)$	0.0330
8	$x^{(8)} = (5.9499 \quad 1.4418 \quad 3.2746)$	0.0113
9	$x^{(9)} = (5.9509 \quad 1.4439 \quad 3.2791)$	0.0050

Clearly that the sequence $\{ x^{(k)} \}$will converge to the solution

$x = (x_1, x_2, x_3) = (\ 5.9509, 1.4439, 3.2791)$

Now, putting the value of (x_1, x_2, x_3) into the equations (19) and (20) we obtain

$$\begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 42.0105 \\ 22.7609 \\ 12.8430 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0.2099 \\ 6.1967 \\ 4.3064 \end{bmatrix}$$

i.e.
$$9y_1 + y_2 = 42.0105$$
$$y_1 + 4y_2 + y_3 = 22.7609$$
$$2y_2 + 4y_3 = 12.8430$$

and
$$9z_1 + z_2 = 0.2099$$
$$7z_1 + 5z_2 + 2z_3 = 6.1967$$
$$2z_2 + 4z_3 = 4.3064$$

The above system are not in the diagonally dominant form. Converting them to the diagonally dominant form gives:

$$9y_1 + y_2 = 42.0105$$

$$y_1 + 4y_2 + y_3 = 22.7609 \tag{23}$$

$$2y_2 + 4y_3 = 12.8430$$

and

$$9z_1 + z_2 = 0.2099$$

$$7z_1 + 5z_2 + 2z_3 = 6.1967 \tag{24}$$

$2z_2 + 4z_3 = 4.3064$
Following similar procedure as before, we obtain

$$y_1 = \frac{1}{9}(42.0105 - y_2)$$
$$y_2 = \frac{1}{4}(22.7609 - y_1 - y_3)$$
$$y_3 = \frac{1}{4}(12.8430 - 2y_2)$$

Starting with the initial approximation $y^{(0)} = (0,0,0)$, the and Gauss- Jacobi method generates a sequence $\{y^{(k)}\}$ shown in table (2). One sees clearly that the sequence $\{y^{(k)}\}$will converge to the solution

$$y = (y_1, y_2, y_3) = (0.7292, 0.5812, 0.9216)$$

Similarly, to solve (24), we choose the initial approximation $z^{(0)} = (0,0,0)$, the and Gauss- Jacobi method generates a sequence $\{z^{(k)}\}$ shown in table (3). The sequence

Table 2: the sequence $y^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss- Jacobi method, with $\varepsilon = 1 \times 10^{-2}$.

k	$y = (y_1, y_2, y_3)$	Error $\lvert y - y_0 \rvert$
1	$y^{(1)}$ = (0.79339 0.99209 1.2105)	1.7547
2	$y^{(2)}$ = (0.68316 0.49113 0.71441)	0.7136
3	$y^{(3)}$ = (0.73882 0.6427 0.9649)	0.2980
4	$y^{(4)}$ = (0.72198 0.56616 0.88911)	0.1090
5	$y^{(5)}$ = (0.73048 0.58932 0.92738)	0.0455
6	$y^{(6)}$ = (0.72791 0.57763 0.9158)	0.0167
7	$y^{(7)}$ = (0.72921 0.58116 0.92165)	0.0070

Table 3: the sequence $z^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss- Jacobi method, with $\varepsilon = 1 \times 10^{-2}$.

k	$z = (z_1, z_2, z_3)$	Error $\lvert z - z_0 \rvert$
1	$z^{(1)}$ = (4.6511 5.223 3.7732)	7.9467
2	$z^{(2)}$ = (4.0708 3.117 1.1616)	3.4048
3	$z^{(3)}$ = (4.3048 3.9149 2.2147)	1.3418
4	$z^{(4)}$ = (4.2161 3.5932 1.8157)	0.5202
5	$z^{(5)}$ = (4.2519 3.7151 1.9766)	0.2050
6	$z^{(6)}$ = (4.2383 3.6659 1.9156)	0.0795
7	$z^{(7)}$ = (4.2438 3.6845 1.9402)	0.0313
8	$z^{(8)}$ = (4.2417 3.677 1.9309)	0.0121
9	$z^{(9)}$ = (4.2426 3.6799 1.9346)	0.0048

$\{z^{(k)}\}$will converge to the exact solution

$$z = (z_1, z_1, z_3) = (4.2426, 3.6799, 1.9346)$$

Hence the solution of given fully fuzzy linear system (17) is

$$\check{x} = \begin{bmatrix} (5.9509,0.7292,4.2426) \\ (1.4439,0.5812,3.6799) \\ (3.2791,0.9216,1.9346) \end{bmatrix}$$

Now, we will attempt to solve example (1) using the Gauss− Seidel method i.e.

$$x_1 = \frac{1}{9}(55 - x_2)$$
$$x_2 = \frac{1}{4}(15 - x_1 - x_3)$$
$$x_3 = \frac{1}{4}(16 - 2x_2)$$

Generating a sequence $X^{(k)}$

$$x_1{}^{(k+1)} = \frac{1}{9}\left(55 - x_2{}^{(k)}\right)$$
$$x_2{}^{(k+1)} = \frac{1}{4}\left(15 - x_1{}^{(k+1)} - x_3{}^{(k)}\right) \ , k = 0,1,2 \ldots\ldots$$
$$x_3{}^{(k+1)} = \frac{1}{4}\left(16 - 2x_2{}^{(k+1)}\right)$$

Taking the $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3{}^{(0)}\right) = (0,0,0)$ we get

$$x_1{}^{(1)} = \frac{1}{9}\left(55 - x_2{}^{(0)}\right) = \frac{55}{9} = 6.1111$$
$$x_2^{(1)} = \frac{1}{4}\left(15 - x_1{}^{(1)} - x_3{}^{(0)}\right) = \frac{20}{9} = 2.2222$$
$$x_3{}^{(1)} = \frac{1}{4}\left(16 - 2x_2{}^{(1)}\right) = \frac{26}{9} = 2.8889$$

i.e. $x^{(1)} = (6.1111,\ 2.2222,\ 2.8889)$ Continuing the process we obtain the results depicted in table (4). One sees clearly that the sequence $\{x^{(k)}\}$ will converges to the

Table 4: the sequence $x^{(k)}, k = 0,1,2, \ldots$ generated by the Gauss-Seidel method, with $\varepsilon = 1 \times 10^{-2}$.

k	$x = (x_1, x_2, x_3)$	Error $\lvert x - x_0 \rvert$
1	$x^{(1)}$ = (6.1111 2.2222 2.8889)	7.1155
2	$x^{(2)}$ = (5.8642 1.5617 3.2191)	0.7786
3	$x^{(3)}$ = (5.9376 1.4608 3.2696)	0.1346
4	$x^{(4)}$ = (5.9488 1.4454 3.2773)	0.0206
5	$x^{(5)}$ = (5.9505 1.443 3.2785)	0.0031

exact solution

$$x = (x_1, x_2, x_3) = (5.9505,1.443,3.2785)$$

Next, Starting with the initial approximation $y^{(0)} = (0,0,0)$, the and Gauss- Seidel method generates a sequence $\{\, y^{(k)}\}$ shown in table (5). Hence the sequence $y^{(k)}$will converges to the exact solution:

$y = (y_1, y_2, y_3) = (0.72835, 0.5811, 0.92021)$

Finally, Starting with the initial approximation $z^{(0)} = (0,0,0)$, the and Gauss- Seidel method generates a sequence $\{z^{(k)}\}$ shown in table (6). Hence from the above results, we find that the value of z is:

$z = (z_1, z_2, z_2) = (4.2413, 3.6807, 1.933)$

Table 5: the sequence $y^{(k)}, k = 0,1,2, ...$ generated by the Gauss-Seidel method, with $\varepsilon = 1 \times 10^{-2}$.

k	$y = (y_1, y_2, y_3)$	Error $\|y - y_0\|$
1	$y^{(1)} = (0.79339 \quad 0.79437 \quad 0.81358)$	1.3865
2	$y^{(2)} = (0.70513 \quad 0.61304 \quad 0.90424)$	0.2211
3	$y^{(3)} = (0.72527 \quad 0.58534 \quad 0.91809)$	0.0369
4	$y^{(4)} = (0.72835 \quad 0.5811 \quad 0.92021)$	0.0056

Table 6: the sequence $z^{(k)}, k = 0,1,2, ...$ generated by the Gauss-Seidel method, with $\varepsilon = 1 \times 10^{-2}$.

k	$z = (z_1, z_2, z_3)$	Error $\|z - z_0\|$
1	$z^{(1)} = (4.6511 \quad 4.0606 \quad 1.7431)$	6.4156
2	$z^{(2)} = (4.2 \quad 3.7376 \quad 1.9046)$	0.5779
3	$z^{(3)} = (4.2358 \quad 3.6882 \quad 1.9293)$	0.0658
4	$z^{(4)} = (4.2413 \quad 3.6807 \quad 1.933)$	0.0101
5	$z^{(5)} = (4.2422 \quad 3.6795 \quad 1.9336)$	0.0015

Hence the solution of the given fully fuzzy linear system of equation up to two decimal places is found to be

$$\tilde{x} = \begin{bmatrix} (5.9505, 1.443, 3.2785) \\ (0.72835, 0.5811, 0.92021) \\ (4.2413, 3.6807, 1.933) \end{bmatrix}$$

6. Conclusion

In this article, Gauss- Jacobi and Gauss- Seidel iterative methods have been used to solve Positive Triangular Fully Fuzzy Linear System where all the coefficient matrix arrays, the right-hand side arrays and the unknowns, are fuzzy numbers. The numerical results have shown to be in a close agreement with the analytical ones. In fact, numerical results show clearly the Gauss- Seidel iterative methods is more efficient than the Gauss-Jacobi for solving Positive Triangular Fully Fuzzy Linear System.

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