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(Co) Homology Theory of *L- infinity* Algebras

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Abstract

In this work we study L-infinity algebras which is generalization of the graded Lie algebras in which an arrangement of anti-symmetric brackets satisfies a generalized Jacobi-identity. We discuss a cyclic and dihedral (co)homology of L-infinity algebras and some of their algebraic properties.

Keywords: Cyclic homology - Graded Algebra - Lie algebra- Tensor product.
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1. Introduction

Although genuine studying of Lie algebras was started by *Nathan Jacobson* in 1950s, their theory of representation in the semi basic case was only recently advanced because of powerful conjectures by *Lusztig*, which was partially proved as from 2007.

The idea of an A_∞-algebra, likewise, called a strongly homotopy associative algebra was presented in [11] and is an estimation of the associative algebra.

From a specific perspective, an associative algebra is just an exceptional instance of a co-differential on the tensor co-algebra of a vector space. A_∞-algebra given by taking the arbitrary co-derivation, especially as examples of A_∞-algebras are associative algebras and differential graded associative algebras (*DGAA*).

L_∞-algebras is defined as strongly homotopy Lie algebras and be generalizing from Lie algebras. The Lie algebra seen as basically an uncommon case of a co-differential on an exterior co-algebra of the vector space, and L_∞-algebras are simply arbitrary co-differentials on this (co)algebra.

A bracket structure was presented in the space of co-chains of an associative algebra in [16]. From a co-algebra perspective, the *Gerstenhaber* bracket ends up being basically the bracket of co-determinations.

In [6] provided an various description of L_∞-structure properties in parts of symmetric brackets and (co)differentials on the symmetric co-algebra.

The meanings of Hochschild (co)homology and cyclic (co)homology of an A_∞-algebra were presented in [18] by *A. Schwarz*. The primary motivation behind this paper was to apply a few developments and study the dihedral (co)homology to L_∞-algebras. It includes definitions of the cyclic (co)homology of Lie algebras and $\mathbb{Z}_2$-graded algebras.

In this paper, we're generalizing a bracket to L_∞-algebras and define the cyclic and dihedral co-homology bracket when the inner product is invariant.

In our paper, we will introduce and study cyclic and dihedral (co)homology of L_∞-algebras and extend some properties of it. Over law, we introduce some facts and relations in the cyclic homology of L_∞-algebras.

The content of this paper is:

In part 1: We are recalling some definitions and background notations in the cohomology theory of L_∞-algebras.

In part 2: We will introduce and study the simplicial and cyclic cohomology of L_∞-algebras and show that the *Connes-Tsygan* exact sequence exists for L_∞-algebra.

In part 3: We discuss the dihedral cohomology and study some properties of L_∞-algebras.

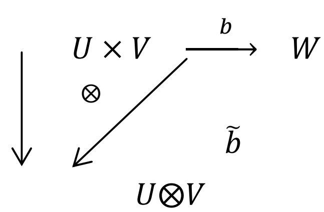
Firstly, we review a few definitions and realities about the graded algebras and lie-algebras.

2. Mathematical Background

In this section, we are recalling background notations in the (co)homology of lie algebras and its homology theory. The main references are [6], [8], [13], [20], [26] and [29].

Definition (1.1):

Let U and V are two vector spaces over $\mathcal{F}$, the tensor product $U \otimes V$ can be defined by the following property $\otimes: U \times V \to U \otimes V$ is the bilinear map with the end goal that for the space W and a bilinear map $b: U \times V \to W$. The map $\tilde{b} : U \otimes V \to W$ is the special linear map , such as $b = \tilde{b} \circ \otimes$, i.e. with ultimate goal that the following diagram commutes:

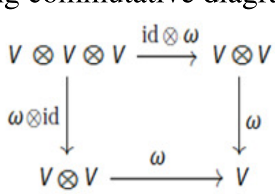


Definition (1.2):

The associative product is the vector space V over $\mathcal{F}$ along with a linear function: $\omega: V \otimes V \to V$, which verifies: $\omega \circ (\omega \otimes id) = \omega \circ (id \otimes \omega)$. The algebra V is unital if there exists a function $f: \mathcal{F} \to V$ defined as unit, such that: $\omega \circ (f \otimes id) = id = \omega \circ (id \otimes f)$.

<u>Note that:[16]</u>

f sends $1_{\mathcal{F}}$ to 1_V and hence $\mathcal{F}$ to $\mathcal{F}1_V \subset V$. The associativity and unitality can be detailed by methods for the corresponding commutative diagrams:



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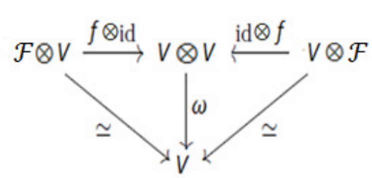


Definition (1.3):

A co-algebra V over the field $\mathcal{F}$ is $\mathcal{F}$-module V equipped with a $\mathcal{F}$-linear map called a co-multiplication $\Delta: V \to V \otimes V$, and the co-unit map $c: V \to \mathcal{F}$, satisfying:

a) (a) co-associativity: $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$,
b) (b) co-unit: $(id \otimes c) \circ \Delta = (c \otimes id) \circ \Delta = id$.

It is a co-commutative co-algebra if moreover $\Delta \circ T = \Delta$, where T the twisting function: $T(\alpha \otimes \beta) = \beta \otimes \alpha$.

Definition (1.4):

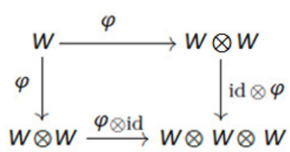
The coassociative co-algebra W is a space over $\mathcal{F}$ along with the linear function $\varphi: W \to W \otimes W$ called co-product, which is co-associative, since verifies:

$$(\varphi \otimes id) \circ \varphi = (id \otimes \varphi) \circ \varphi \; . \tag{1}$$

The co-associative co-algebra W be called co-unital if there exists a map $g: W \to \mathcal{F}$, called co-unit, to such an extent that:

$$(g \otimes id) \circ \varphi = id = (id \otimes g) \circ \varphi. \tag{2}$$

Co-unitality and co-associativity can be figured by methods for the accompanying commutative diagrams:



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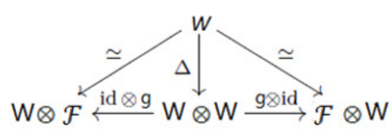


Definition (1.5):

The category with a bifunctor $\otimes: C \times C \to C$ called the monoidal category C and the object I called unit, satisfying associativity and unity requirements.

Examples (1.6):[28]

1. The category of the space V over $\mathcal{F}$ with a tensor product $\otimes$ and the unit $I = \mathcal{F}$ is called monoidal category.
2. A category $E(C) = [C, C]$ of endo-functors in C, whose objects are functors: $C \to C$ and morphisms are natural transformations is called monoidal category.

Definition (1.7):

The monoidal category $(C,\otimes, I)$ have monoid is object $\mathcal{L}$ with the morphisms $\gamma: \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ and $i: I \to \mathcal{L}$, satisfy the unity and the associative requirements.

Example (1.8):[21]

The (associative) unital algebra with the multiplication γ and the unit i is a monoid in $(V,\otimes,\mathcal{F})$. The monoid can't be characterized in a discretionary category C. In any case, it is consistently conceivable to characterize a monoid in the category $E(C)$, which is a monoidal category. A monoid in $E(C)$ is called a monad in C.

Definition (1.9):

The operad in classical is given by a multi category $\mathcal{C}$ which is consists of:

1. A set $\mathcal{C}_0$ of objects denoted by , $a_1, a_2, \cdots$.

2. For any $n \in \mathbb{N}, a_1, \cdots, a_n; a$ a set $Hom(a_1, \cdots, a_n; a)$ of morphisms.
3. A composition map $\gamma_{k_1,\ldots,k_n}$:

$$Hom(a_1, \cdots, a_n; a) \times Hom(a_{11}, \cdots, a_1k_1; a) \times \cdots \times Hom(a_{n1}, \cdots, a_nk_n; a)$$
$$\to Hom(a_{n1}, \cdots, a_nk_n; a)$$
$$(\theta; \theta_1, \cdots, \theta_n) \mapsto \theta \circ (\theta_1, \cdots, \theta_n).$$

4. For any element a, the identity morphism $1_a \in \mathrm{Hom}(a, a)$, with the end goal that composition is associativity and has identity.
5. 5) An operad is an example of a multi-category with a unique object (see [27]).

Definition (1.10):

A non-symmetric operad $\mathfrak{L}$ contains of:

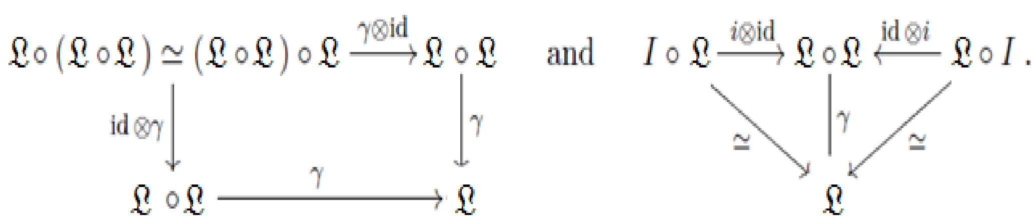
1) A sequence $(\mathfrak{L}(n))_{n\in\mathbb{N}}$ of sets, whose components are called unique n-ary operations of $\mathfrak{L}$.
2) For all integer $n, k_1, \cdots, k_n$, the map $\gamma_{k_1,\ldots,k_n}$:

   $\mathfrak{L}(n) \times \mathfrak{L}(k_1) \times \cdots \times \mathfrak{L}(k_n) \to \mathfrak{L}(k_1 + \cdots + k_n)$
   The composition is defined by the rule: $(\theta; \theta_1, \cdots, \theta_n) \mapsto \theta \circ (\theta_1, \cdots, \theta_n)$.
3) For each $1 \in \mathfrak{L}(1)$ (identity), satisfying the properties:

$$\theta \circ \left(\theta_1 \circ \left(\theta_{1,1}, \cdots, \theta_{1,k_1}\right), \cdots, \theta n \circ \left(\theta_{n,1}, \cdots, \theta_{n,k_n}\right)\right) =$$
$$\left(\theta \circ (\theta_1, \cdots, \theta_n)\right) \circ \left(\theta_{1,1}, \cdots, \theta_{1,k_1}, \cdots, \theta_{n,1}, \cdots, \theta_{n,k_n}\right), \qquad \text{associative}$$
$$\theta \circ (1, \cdots, 1) = \theta = 1 \circ \theta. \qquad unity$$

Definition (1.11):

The operad be the monad in the category $\mathcal{C}$, i.e. the monoid in the category $\mathrm{E}(\mathcal{C})$ by monoidal structure $\circ$. Specifically, the operad $\mathcal{L}$ is the object in $E(\mathcal{C})$, i.e. the functor $\mathfrak{L} : \mathcal{C} \to \mathcal{C}$ including the maps $\gamma: \mathfrak{L} \circ \mathfrak{L} \to \mathfrak{L}$ and $\mathrm{i}: \mathrm{I} \to \mathfrak{L}$, which are normal transformations, satisfy the unity and the associativity requirements provided by a commutative diagrams below:



In [15], for two endo-functors $\mathfrak{L}, \mathcal{Q} \in \mathrm{E}(\mathcal{C})$, a composition $\circ$ is defined obviously by: $(\mathfrak{L} \circ \mathcal{Q})(\mathcal{C}) = \mathfrak{L}(\mathcal{Q}(\mathcal{C}))$ and $(\mathfrak{L} \circ \mathcal{Q})(\ell) = \mathfrak{L}(\mathcal{Q}(\ell))$, for a vector space $\mathcal{C}$ and a linear map ℓ. Often it is possible to describe additional operations on endo-functors in V, a tensor product and a direct sum, namely by:

$(\mathfrak{L} \otimes \mathcal{Q})(\mathcal{C}) = \mathfrak{L}(\mathcal{C}) \otimes \mathcal{Q}(\mathcal{C})$, $(\mathfrak{L} \otimes \mathcal{Q})(\ell) = \mathfrak{L}(\ell) \otimes \mathcal{Q}(\ell)$,
Respectively: $(\mathfrak{L} \oplus \mathcal{Q})(\mathcal{C}) = \mathfrak{L}(\mathcal{C}) \oplus \mathcal{Q}(\mathcal{C})$, $(\mathfrak{L} \oplus \mathcal{Q})(\ell) = \mathfrak{L}(\ell) \oplus \mathcal{Q}(\ell)$,

Definition (1.12): [24]

Just as well as associative algebras be the algebras over the non-symmetric operad constructed naturally, A_∞-algebras can be seen as algebras over a obviously non-symmetric operad A_∞, what we are going to describe.

Let $(A, m_1, m_2, \cdots), m_k \in Hom(A^{\otimes k}, A), degm_k = k-2$, be A_∞-algebra. For $n = 1$, define the relation by:

$$\sum_{\substack{p+q+r=n \\ p+1+r=k \\ k,q\geq 1}} (-1)^{p+qr} m_k \left(\underbrace{id, \cdots, id}_{(p)}, m_q, \underbrace{id, \cdots, id}_{(r)} \right) = 0 \ , n \geq 1, \quad (3)$$

reads about $m_1 \circ m_1 = 0$, sothat $d := -m_1 \in End_{-1}(A)$ given a graded vector space A with the chain complex structure. Consequently, $A^{\otimes n}$ be the chain complex for the differential $d_{A^{\otimes n}} = \sum_{p+1+r=n} id^{\otimes p} \otimes d \otimes id^{\otimes r}$. This entails that $Hom(A^{\otimes n}, A)$ is a chain complex of the differential $\partial = [d, -] = [-m_1, -]$. Therefore,

$$\partial m_n = -m_1(m_n) + (-1)^{n-2} m_n \left(\sum_{p+1+r=n} id^{\otimes p} \otimes m_1 \otimes id^{\otimes r} \right), \quad (4)$$

and for $n \geq 2$, read

$$\partial m_n = \sum_{\substack{p+q+r=n \\ p+1+r=k \\ \text{k,q} \geq 2}} (-1)^{p+qr} m_k \left(id^{\otimes p} \otimes m_q \otimes id^{\otimes r} \right). \quad (5)$$

As the A_∞-algebra be a chain complex (A, d) endowed with the operations $m_n \in Hom(A^{\otimes n}, A)$ of degree $n-2$, $n \geq 2$, that verifies the last relations, the appropriate operad be the non-symmetric deferential graded operad A_∞.

Definition (1.13):

A Hopf algebra $\mathcal{H} = (\mathcal{H}, \mu, u, \Delta, c)$ over $\mathcal{F}$ be the $\mathcal{F}$-module which is, simultaneous, the unital $\mathcal{F}$-algebra are μ (multiplication) and u (unit map) and the co-unital k-coalgebra are Δ (co-multiplication) and c (co-unit map), and they fulfill the conditions:

(i) Multiplication be a map of coalgebra and co-muitiplication be a map of algebra.
(ii) The map of unit be a co-algebra map and the map of counit be an algebra map.

The composite map: $c \circ u: \mathcal{F} \to \mathcal{F}$ is id_k. The map $S: \mathcal{H} \to \mathcal{H}$ with the conditions: $S^2 = id$ and $\mu \circ (id \otimes S) \circ \Delta = \mu \circ (S \otimes id) \circ \Delta = id$, is called the antipodal map.

Definition (1.14):

The graded vector space be the vector space E combined with the decomposition $E \cong \bigoplus_{p\in\mathbb{Z}} E_p$ for the vector spaces family $\{E_p\}_{p\in\mathbb{Z}}$. The element $e \in E_p$ can be called homogeneous of degree p and rewrite $|e| = p$.

It is presumed that all vector spaces are over the fixed ground field $\mathcal{F}$ characteristically zero. We often define the graded vector spaces through E and H and by the homogeneous elements $e_1, \cdots, e_n \in E$.

The linear map $f: E \to H$ is considered homogeneous degree p when there is $p \in \mathbb{Z}$ such that: $f(E_n) \subset H_{n+p}$ $\forall\ n \in \mathbb{Z}$. We would denote the homogeneous linear maps $E \to H$ as a vector space with degree p by $Hom_p(E, H)$, and also a graded vector space $\bigoplus_{p\in\mathbb{Z}} Hom_p(E, H)$ by $Hom(E, H)$. The elements in $Hom_0(E, H)$ are also called degree preserving.

A standard grading exists on direct sum of E and H denoted as $(E \oplus H)_p = E_p \oplus H_p$. The isomorphism:

$$E \otimes H \cong \bigoplus_{p\in\mathbb{Z}} \bigoplus_{i+j=p} (E_i \otimes H_j), \quad (6)$$

makes it possible to determine the grading on $E \otimes H$ by $(E \otimes H)_p = \oplus_{i+j=p} (E_i \otimes H_j)$.

That leads to the grading on $E^{\otimes n} = \otimes_{i=1}^{n} E$ denoted by:

$$(E^{\otimes n})_p = \oplus_{i_1+\cdots+i_n=p} E_{i_1} \otimes \cdots \otimes E_{i_n}. \tag{7}$$

We define by $(\rho_{E,H})$ the linear degree preserving map:

$$\rho_{E,H} : E \otimes H \to E \otimes H \quad , \quad e \otimes h \mapsto (-1)^{|e||h|}\, e \otimes h \,. \tag{8}$$

If $f \in Hom(E,H)$ and $g \in Hom(E',H')$ homogeneous for the graded vector spaces E' and H', can be defined the linear map $f \otimes g : E \otimes E' \to H \otimes H'$ by:

$$(f \otimes g)(e \otimes e') = (-1)^{|f||g|} f(e) \otimes g(e') \tag{9}$$

for $e \in E$, $e' \in E'$ homogeneous and $|f \otimes g| = |f| + |g|$.

That generalizes to the tensor products with three or more vector spaces clearly, and we're just abbreviating: $f \otimes \cdots \otimes f : E^{(\otimes n)} \to H^{(\otimes n)}$ to $f^{\otimes n}$.

To create such functions, (9)implies:

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}\, (f' \circ f) \otimes (g' \circ g), \tag{10}$$

since f' and g' represent homogeneous linear functions of fields H and H'.

While operating within the graded vector spaces, The rule for signs generally is that when p and q are the degree for two grades symbols, adjust its order within the equation, the sign $(-1)^{|p||q|}$ must be in there, which known as a *Koszul sign* convention.

Definition (1.15):

The symmetric group is denoted by S_n, the all permutations group of the set $\{1, \cdots, n\}$, and also by $s_i \in S_n$ for all $1 \le i \le n-1$ transposing with $s_i(i) = i+1$ and $s_i(i+1) = i$. Two standard linear right S_n-actions on $E^{\otimes n}$ occur. These are given on a generated subset $\{s_1, \cdots, s_{n-1}\} \subset S_n$ by:

$$\Omega'(s_i)(e_1 \otimes \cdots \otimes e_n) = (-1)^{|e_1||e_{i+1}|} e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_n \,, \tag{11}$$

$$\tau'(s_i)(e_1 \otimes \cdots \otimes e_n) = -(-1)^{|e_1||e_{i+1}|} e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_n. \tag{12}$$

Since a graded symmetric Ω' and a graded anti-symmetric τ' action of $S_n$on $E^{\otimes n}$.

<u>Note that:</u> [18]

For all $\sigma \in S_n$, then $\Omega'(\sigma)$ is the degree preserving as:

$$\Omega'(\sigma)(e_1 \otimes \cdots \otimes e_n) = \pm e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}. \tag{13}$$

Then we can denote the sign in (1.13) by $\Omega(\sigma, e_1, \cdots, e_n)$ and likewise by $\tau(\sigma, e_1, \cdots, e_n)$ the sign as:

$$\tau'(\sigma)(e_1 \otimes \cdots \otimes e_n) = \tau\,(\sigma, e_1, \cdots, e_n) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}. \tag{14}$$

We can abbreviate $\Omega(\sigma, e_1, \cdots, e_n)$ and $\tau(\sigma, e_1, \cdots, e_n)$ to $\Omega(\sigma)$ and $\tau(\sigma)$.

Remark (1.16):

Let $U_S \subset E^{\otimes n}$ be a graded subspace which extended by the all elements of the formula:

$$e_1 \otimes \cdots \otimes e_n - \Omega'(\sigma)(e_1 \otimes \cdots \otimes e_n). \tag{15}$$

for $\sigma \in S_n$. The space $S^n(E) = E^{\otimes n}/U_S$ is classified as $n^{\underline{th}}$ symmetric exponent of E. Likewise, the $n^{\underline{th}}$ exterior exponent of E defines it as a quotient of $E^{\otimes n}$ through a graded sub-space which extended through the elements of the formula:

$$e_1 \otimes \cdots \otimes e_n - \tau'(\sigma)(e_1 \otimes \cdots \otimes e_n). \tag{16}$$

$\forall\ \sigma \in S_n$, it's denoted by $\gamma^n E$.

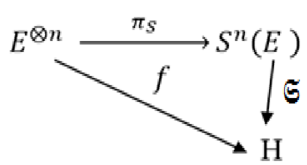
The n-linear function $f: E^n \to H$ is symmetric when:

$$f(e_1 \otimes \cdots \otimes e_n) = \sigma(\sigma) f\big(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}\big), \quad \forall\ \sigma \in S_n. \tag{17}$$

holds. We can easily write that conveniently as: $f \circ \Omega'(\sigma) = f$. Also, f is called anti-symmetric when: $f \circ \tau'(\sigma) = f\ ,\ \forall\ \sigma \in S_n$

Proposition (1.17):

Let the symmetric linear function is given by the function $f: E^{\otimes n} \to H$. Where the linear function $\mathfrak{S}: S^n(E) \to H$ is unique such that the diagram below satisfies:



Proof:

Since the map f is symmetric, it is disappearing on the U_S generators and the π_S factors out to the linear function $\mathfrak{S}: S^n(E) \to H$ so that the above diagram satisfy. This function is unique as π_S be surjective.

Remark (1.18):

By proposition (1.17) we can get:

1. Since a symmetric S_n-action on $E^{\otimes n}$ is a degree preserving, where $S^n(E)$ takes the essential grading from $E^{\otimes n}$ such that π_S be a degree preserving. So it instantly happens if f is homogeneous, and that is the map $\mathfrak{S}$ and $|\mathfrak{S}| = |f|$.
2. Since π_S is symmetric via construction of $S^n(E)$, It brings about isomorphism between subspace of $Hom(E^{\otimes n}, H)$ consists of all the symmetric maps and $Hom(S^n(E), H)$.
3. The analog of this applies to $\gamma^n E$ and inducing the isomorphism between the subspace $Hom(E^{\otimes n}, H)$ of all anti-symmetric maps and $Hom(\gamma^n E, H)$.
4. The element in $E^{\otimes n}$ is said to be symmetric when it is being an invariance under the symmetric S_n-action on $E^{\otimes n}$. We say that $S^n(E)$ be isomorphic to a subspace of $E^{\otimes n}$of all symmetric elements. In fact, allowing $e_1\, \gamma\ \ldots\ \gamma\, e_n$ indicate an image of $e_1 \otimes \cdots \otimes e_n$ under π_S, linear map $\mathfrak{S}: S^n(E) \to E^{\otimes n}$:

$$e_1 \gamma \cdots \gamma e_n \longmapsto \frac{1}{n!}\sum_{\sigma \in S_n} \Omega(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}. \tag{18}$$

be well-defined:

$$\pi_S \circ \mathfrak{S} = idS^n(E) \text{ and } \mathfrak{S} \circ \pi_g = \frac{1}{n!}\sum_{\sigma \in S_n} \Omega'(\sigma). \tag{19}$$

5. Since the latter be the projection of $E^{\otimes n}$ onto that sub-space. The similar statement holds for $\gamma^n E$ clearly.

For $n \in \mathbb{Z}$, a graded vector space $E[n]$ is defined to be a vector space E with a grading identified by $E[n]_p$=E_{p+n} . We can denote an identity map on E by $\downarrow^n : E \longrightarrow E[n]$, which turns into the linear isomorphism of degree $-n$, and by its opposite $\uparrow^n$. The abbreviation is $\downarrow^1$ and $\uparrow^1$to $\downarrow$ and $\uparrow$ respectively, $(\downarrow^{\otimes k})^{-1} = (-1)^{\frac{k(k-1)}{2}} \uparrow^{\otimes k}$as a consequence of (10).

Proposition (1.19):

For all $\sigma \in S_n$, $\Omega'(\sigma) \circ \downarrow^{\otimes n} = \downarrow^{\otimes n} \circ \tau'(\sigma)$. Then there's the degree preserving isomorphism: $S^n(E[1]) \cong (\gamma^n E)[n]$.

Remark (1.20):

Let A and B are two graded associative algebras. Then can define the multiplication by: $(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'$, for $a, a' \in A$, $b, b' \in B$ homogeneous given $A \otimes B$ into the graded associative algebra. Where both A and B are commutative and unital, so it is $A \otimes B$.

Example (1.21):

The tensor algebra is denoted by $T(E) = \bigoplus_{n \geq 0} E^{\otimes n}$. It holds a multiplication concluded by a canonical isomorphism: $E^{\otimes r} \otimes E^{\otimes s} \cong E^{\otimes (r+s)}$, makes it inside the unitary associative algebra. A grading on $T(E)$ concluded with a grading on $E^{\otimes n}$ is denoted by:

$$T(E)_p = \bigoplus_{i_1 + \cdots + i_n = p} E_{i_1} \otimes \cdots \otimes E_{i_n}, \tag{20}$$

which called the interior grading. Otherwise, $T(E)$ holds a grading provided by $(E) = \bigoplus_{n \geq 0} E^{\otimes n}$, which called the exterior grading. Unless otherwise specified, We consider $T(E)$ always to bear its interior grading. The interior and exterior grading makes $T(E)$ into the graded algebra.

Definition (1.22):

The graded Lie algebra is given by the graded vector space L in combined with an anti-symmetric degree preserving linear function $[\cdot,\cdot]: L \otimes L \longrightarrow L$ calls the Lie bracket that satisfies the Jacobi identity:

$$[a, [b, c]] = [[a, b], c] \pm (-1)^{|a||b|}[b, [a, c]] = 0\,, \tag{22}$$

$\forall\ a, b, c \in L$ homogeneous.

If L remains ungraded, we get back the regular definition of the Lie algebra. Notice it is none other than $[x,\cdot]$ to be a derivative of graded algebra $(L, [\cdot,\cdot])$.

Example (1.23):

For the graded associative algebra V, the graded commutator $[\cdot,\cdot]: V \otimes V \to V$ can be defined by $[a, b] = ab - (-1)^{|a||b|} ba$, $\forall\, a, b \in V$ homogeneous, this take V into the graded Lie algebra.

Specially, $gl(E) = Hom(E, E)$ turn into a graded Lie algebra. When E is a graded algebra by itself, we could verify that $Der(E) \subset gl(E)$ is the Lie sub-algebra.

Definition (1.24):

The differential graded Lie algebra is the differential graded algebra such that a fundamental algebra is a graded Lie algebra.

Example (1.25):

Let the graded Lie algebra L and the degree of one element $a \in L$, such that $\frac{1}{2}[a, a] = 0$. Then $d = [a,\cdot]$ satisfy $d^2 = 0$ by *Jacobi-identity* and $(L, [\cdot,\cdot], d)$ is the $DGLA$. Particularly, for (E, ∂) the DG vector space, it makes $(gl(E), [\cdot,\cdot], [\partial,\cdot])$ canonically in the graded Lie algebra as: $\frac{1}{2}[\partial, \partial] = \partial^2 = 0$.

Definition (1.26):

For the graded Lie algebra $(L, [\cdot,\cdot], d)$, the element of *Maurer–Cartan* is the element $a \in L$ of the degree one such that: $(a) + \frac{1}{2}[a, a] = 0$, this equation can be called a *Maurer–Cartan* equation.

Example (1.27):[5]

Let $(L, [\cdot,\cdot], d = [a,\cdot])$ be like in example (1.25). For $b \in L$ of degree one, then we can have: $\frac{1}{2}[\mathrm{a} + \mathrm{b}, \mathrm{a} + \mathrm{b}] = 0$, iff b satisfy a *Maurer–Cartan* equation. For $0 \le i \le n$, the $(i, n-i)$-unshuffle be the permutation $\sigma \in S_n$ satisfies $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. The set of all $(i, n-i)$-unshuffles denoted by $Sh_{i,n-i}^{-1} \subset S_n$. By usage of the anti-symmetry of Lie bracket, we can rewrite (21) like as:

$$\sum_{\sigma \in Sh_{2,1}^{-1}} a(\sigma)[[a_{\sigma(1)}, a_{\sigma(2)}], a_{\sigma(3)}] = 0 \tag{23}$$

for all $a_1, a_2, a_3 \in L$ homogeneous.

Lemma (1.28):

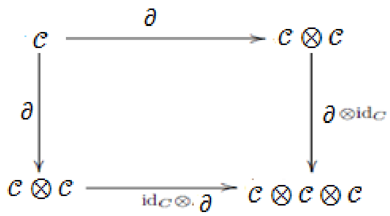
For each element $\sigma \in S_n$ includes the unique decomposition: $\sigma = \tau(\alpha, \beta)$ for each $i \in \{0, \cdots, n\}$, where $\tau \in Sh_{i,n-i}^{-1}$ and $(\alpha, \beta) \in S_i \times S_{n-i}$. Hence, $S_i \times S_{n-i}$ is considered to be the subgroup of S_n.

Proof:

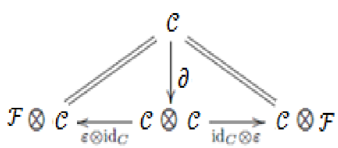
τ must be the unique $(i, n-i)$-unshuffle like that $\{\tau(1), \cdots, \tau(i)\} = \{\sigma(1), \cdots, \sigma(i)\}$ and $\{\tau(i+1), \cdots, \tau(n)\} = \{\sigma(i+1), \cdots, \sigma(n)\}$. Then we are having $\tau^{-1}\sigma \in S_i \times S_{n-i}$.

Definition (1.29):

The graded co-algebra $(\mathcal{C}, \partial)$ is the graded vector space $\mathcal{C}$ combined with the degree preserving linear function $\partial: \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is called the coproduct. When the following diagram:



commutes, $\mathcal{C}$ called co-associative. And $\mathcal{C}$ can be called co-unital when there exists the degree preserving linear function $\varepsilon: \mathcal{C} \to \mathcal{F}$ so that the diagram below:



holds. A map ε is defined as the co-unit of $\mathcal{C}$. When $\tau_{\mathcal{C},\mathcal{C}} \circ \partial = \partial$, then $\mathcal{C}$ is called co-commutative. The map of linear degree preserving $f: \mathcal{C} \to D$ between co-algebras$(\mathcal{C}, \partial_{\mathcal{C}})$ and (D, ∂_D) be called the homomorphism of co-algebras when:

$$(f \otimes f)\partial_C = \partial_D \circ f. \tag{23}$$

If $\mathcal{C}$ and D are the co-unital with the co-units ε and μ, respectively, when f satisfies: $\mu \circ f = \varepsilon$, since f is called the homomorphism of co-unital co-algebras. For the co-associative co-algebra $(\mathcal{C}, \partial)$ and $n \in \mathbb{N}$, we can define the iterated co-product $\partial^n: \mathcal{C} \to \mathcal{C}^{\otimes(n+1)}$ by $\partial^0 = id_{\mathcal{C}}$ and $\partial^n = (\partial \otimes id_{\mathcal{C}} \cdots \otimes id_{\mathcal{C}})\partial^{n-1}$ for $n \geq 1$. It is appropriate to then using *Sweedler-notation* and for $a \in \mathcal{C}$, can write $\partial^{n(a)} \in \mathcal{C}^{\otimes(n+1)}$ as:

$$\partial^{n(a)} = \sum a_{(1)} \otimes \cdots \otimes a_{(n+1)}. \tag{24}$$

For example, by the notation, the condition of $\mathcal{C}$ to be co-commutative being:

$$\sum a_{(1)} \otimes a_{(2)} = \sum (-1)^{|a_{(1)}||a_{(2)}|} a_{(2)} \otimes a_{(1)}, \quad \forall\, a \in \mathcal{C}. \tag{25}$$

Definition (1.30):

The A_∞-algebra is the graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E^i$ with the maps $m_n : E^{\otimes n} \to E$, $\forall\, n \in \mathbb{N}$, of degree $|m_n| = 2 - n$, that satisfies the equations:

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t} \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0, \quad n \quad \big(SI(n)\big) \tag{26}$$

For example:

I. $SI(1)$ means that $m_1 \circ m_1 = 0$.
II. $SI(2)$ says that m_1 is a derivation for m_2.
III. $SI(3)$ means that m_2 is associative up to the homotopy m_3:

$m_2 \circ (id \otimes m_2) - m_2 \circ (m_2 \otimes id) = \delta(m_3)$
where δ is the differential of $Hom(E^{\otimes 3}, E)$ induced by m_1.

Definition (1.31):

A morphism of A_∞-algebras$g:(E\ ,m^E)\to(H,m^H)$ is a collection of maps:

$g_n: E^{\otimes n}\to H,\ \forall n\in\mathbb{N}$, of degree:$|g_n| = 1-n$ **,** satisfying the equations:

$$\sum_{r+s+t=n}(-1)^{r+st}g_{r+1+t}\circ\left(id^{\otimes r}\otimes m_s^E\otimes id^{\otimes t}\right)$$

$$=\sum\nolimits_{n_1+\cdots+n_q=n}\pm\, m_q^H\circ\left(g_{n_1}\otimes\cdots\otimes g_{n_q}\right),\qquad n\in\mathbb{N}\left(MI(n)\right) \tag{27}$$

Definition (1.32):

L_∞-algebra is the graded vector space L in combination with anti-symmetric linear maps: $l_k: L^{\otimes k}\to L$ identified as higher brackets of the degree:$|l_k| = 2-k$ for $1\le k<\infty$ such a generalized *Jacobi-identity*:

$$\sum\nolimits_{i+j=n+1}\sum\nolimits_{\sigma\in Sh_{2,1}^{-1}}(-1)^{i(j-1)}a(\sigma)\ ,\qquad l_j(l_i(a_1,\cdots,a_i),a_{i+1},\cdots,a_n)\ =\ 0, \tag{28}$$

holds $\forall\ n\ge 1$ and $a_1,\cdots,a_n\in L$ homogeneous. And we then can name the set $\{l_k|\ 1\le k<1\}$ the L_∞-structure on L. By rewriting (28) for $n=1,2,3$ gets:

$$\sum\nolimits_{i+j=n+1}\sum\nolimits_{\sigma\in Sh_{2,1}^{-1}}(-1)^{i(j-1)}a(\sigma),\qquad l_j(l_i(a_1,a_2),a_3)\ =\ 0. \tag{29}$$

for all $a_1,a_2,a_3\in L$ homogeneous.

However, two first equations can be summed up by stating that l_1 be the differential on a graded algebra (L,l_2), This reveals the third equation identifying the defect of the *Jacobi-identity* in (L,l_2). Specifically, the L_∞-algebra with $l_k=0$ for $k\ge 3$ is none other than a *DGLA*. If L is contained within degree zero, $l_k=0$ for $k\ne 2$ for reasons of degree and (L,l_2) be the ungraded Lie algebra.

Definition (1.33):

If L and L' are the L_∞-algebras with the L_∞-structures $\{l_k\}_{k\in N}$ and $\{l_k'\}_{k\in N}$. Then the degree preserving linear function $g: L\to L'$ is defined as the strict L_∞-algebra homeomorphism which satisfies that:

$$g\circ l_k = l_l'\circ g^{\otimes k},\qquad \forall\ 1\le k<\infty. \tag{30}$$

The different characterize of the L_∞-algebras would after that give rise to more general concept of the L_∞-algebra homeomorphisms.

Definition (1.34):

For the non-graded Lie algebra $(f,[\cdot,\cdot])$, the description for function f on the non-graded vector space A is the Lie algebras homomorphism $\sigma: f\to fl(A)$. As for such σ, an Lie algebra cohomology together with the values in A be a cohomology of a *Chevalley–Eilenberg* co-chain complex $(\bigoplus_{n\ge 0} Hom(\Lambda^n f,A),\tau)$ where to the anti-symmetric linear map $\vartheta: f\otimes n\to A$, we define $\tau\vartheta$ by:

$$\tau\vartheta(v_1,\ldots,v_{n+1}) =$$
$$\sum\nolimits_{i=1}^{n+1}(-1)^{i+1}\sigma(v_i)(\vartheta(v_1,\cdots,\hat{v}_i,\cdots,v_{n+1}))+$$
$$\sum\nolimits_{1\le j<k\le n+1}(-1)^{j+k}\times\vartheta([v_j,v_k],v_1,\cdots,\hat{v}_j,\cdots,\hat{v}_k,\cdots,v_{n+1}). \tag{31}$$

for $v_1,\cdots,v_{n+1}\in f$. The elements with ˆ are going to be omitted from the sums above.

Theorem (1.35):

For each $n \geq 1$, a one-to-one correspondence exists between the L_∞-algebras L so that $L_k = 0$ for $k \neq -n$, 0 and $l_1 = 0$ and quadruples (f, A, τ, l_{n+2}) Comprised of the Lie algebra f, the representation τ of f on the vector space A and the $(n+2)$ -cocycle l_{n+2}.

Proof:

For the L_∞-algebra $L = L_0 \oplus L_{-n}$ with $l_1 = 0$, the all brackets excepting l_2 and l_{n+2} must vanish by reasons of degree. Also, l_2 must be vanishes on $\Lambda^2 L_{-n}$ and l_{n+2} could only be nontrivial on $\Lambda^{n+2} L_0$ shown in L_{-n} using relation:

$$\Lambda^n(A \oplus B) \cong \bigoplus_{p+q=n} \Lambda^p A \otimes \Lambda^q B. \tag{32}$$

By decomposing l_2 into the linear maps $[\cdot,\cdot] : \Lambda^2 L_0 \to L_0$ and $\sigma: L_0 \otimes L_{-n} \to L_{-n}$. Then it is the case of computation to prove that l_2 since l_2 satisfies:

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh_{2,1}^{-1}} (-1)^{i(j-1)} a(\sigma) \ ,$$

$$l_j(l_i(a_1, \cdots, a_i), a_{i+1}, \cdots, a_n) = 0 \tag{33}$$

amounts to $(L_0, [\cdot,\cdot])$ to be the Lie algebra, the representation of L_0 on L_{-n} is given by ρ and the co-cycle is L_{n+2} .

Definition (1.36):

The representations up to the homotopy of L_∞-algebras be frequently described as regards the anti-symmetric maps, by starting with a description that holds a symmetrical view of last section.

While we can see the proof of equivalence between the definitions is direct computation. Then can shows that the representations up to the homotopy are none other than weak L_∞-algebra homeomorphisms to $gl(E)$ for the differential graded vector space E, the characterization because of *Lada and Markl*, the representations up to a homotopy were outlined as a differentials on $S(L[1]^*) \otimes E$.

Definition (1.37):

Let L be the L_∞-algebra with L_∞-structure $\{lk | 1 \leq k < \infty\}$. Then the representation up to the homotopy of L on a *DG* vector space E is the linear function $\rho: S(L[1]) \otimes E \to E$ with degree one which satisfies:

$$\rho(d \otimes id_E) + \rho(id_S \otimes \rho)(\Delta_S \otimes id_E) = 0. \tag{34}$$

Lemma (1.38):

Let (L', l'_2, l'_1) be the differential graded Lie algebra and $y' = y'_1 + y'_2$ be the corresponding linear map of degree one $\bar{S}(L'[1]) \to L'[1]$. For the linear degree preserving map $f: \bar{S}(L[1]) \to L'[1]$, the produced homeomorphism of co-algebras $\tilde{f}: \bar{S}(L[1]) \to \bar{S}(L'[1])$ is the homeomorphism of L_∞-algebras iff:

$$f \circ \bar{d} = y'_2 \circ f + \tfrac{1}{2} y'_2 (f \otimes f) \bar{\Delta}_s \ . \tag{35}$$

That is only the situation if the linear map of degree one $\rho: \bar{S}(L[1]) \to L'$ described by $f(a) = (-1)^{|a|+1} \downarrow \rho(a)$ satisfy:

$$\rho \circ \bar{d} + l'_1 \circ \rho + \tfrac{1}{2} l'_2 (\rho \otimes \rho) \bar{\Delta}_s = 0 \ . \tag{36}$$

Theorem (1.39):

The one-to-one correspondence be existing among the representations of L on E and the pairs $(\partial, \tilde{f})$, since ∂ be the differential on E and $\tilde{f}: S(L[1]) \to S(gl(E)[1])$ is the L_∞-algebras homeomorphism. Hence, $gl(E)$ Holds the structure of differential graded Lie algebra caused by ∂.

Proof:

Accordingly $Hom(S(L[1]) \otimes E, E) \cong Hom(S(L[1]), gl(E))$, the linear map of degree one $\rho: S(L[1]) \otimes E \to E$ could be disassembled into the linear maps of degree one $\tilde{\rho}: \bar{S}(L[1]) \to gl(E)$ and $\rho_0 : \mathcal{F} \to gl(E)$, this last one that is the equivalent to an option of the degree one element $\partial = \rho_0(1_{\mathcal{F}}) \in gl(E)$. If we display that under this definition ρ satisfy (34) be equivalent to $\partial^2 = 0$ and $\tilde{\rho}$ satisfy (36), the assumption followed by Lemma (1.38). For $a \in \bar{S}(L[1])$ homogeneous,

$$\frac{1}{2}([.,.](\tilde{\rho} \otimes \tilde{\rho})\bar{\Delta}_s)(a) = \frac{1}{2}\sum(-1)^{|a_{(1)}|}[\tilde{\rho}(a_{(1)}), \tilde{\rho}(a_{(2)})] = \frac{1}{2}\sum(-1)^{|a_{(1)}|}\tilde{\rho}(a_{(1)}) \circ \tilde{\rho}(a_{(2)}) + (-1)^{|a_{(1)}||a_{(2)}|+|a_{(2)}|}\tilde{\rho}(a_{(2)}) \circ \tilde{\rho}(a_{(1)}) = \frac{1}{2}\sum \rho(id_s \otimes \rho)\left(a_{(1)} \circ a_{(2)}\right) + (-1)^{|a_{(1)}||a_{(2)}|}a_{(2)} \otimes a_{(1)}, .) = \rho(id_s \otimes \rho)(\bar{\Delta}_s(a), .). \quad (38)$$

By the co-commutativity of $\bar{S}(L[1])$ and,

$$[\partial, \tilde{\rho}(a)] = \partial \circ \tilde{\rho}(a) - (-1)^{|\tilde{\rho}(a)|}\tilde{\rho}(a) \circ \partial = \rho_0(1) \circ \tilde{\rho}(a) + (-1)|a|\tilde{\rho}(x) \circ \rho_0(1) = \rho(id_S \otimes \rho)(1 \otimes a + a \otimes 1, \cdot). \quad (39)$$

Such as $(\tilde{\rho} \circ d)(a) = \rho(d \otimes id_E)(a, \cdot)$, $\tilde{\rho}$ satisfy (36) be equivalent to (34) keeping $\bar{S}(L[1]) \otimes E$. Also, we have:

$$(\rho(d \otimes id_E) + \rho(id_S \otimes \rho)(\bar{\Delta}_s \otimes id_E))(1, .) = \rho(1, \rho(1, .)) = \partial^2, \quad (40)$$

which finishes proof as: $S(L[1]) = \mathcal{F} \oplus \bar{S}(L[1])$.

Example (1.40):[11]

Let (E, ∂) be the differential graded vector space. The L_∞-algebras $0: L \to gl(E)$ have a trivial strict homeomorphism. The produced representation $S(L[1]) \otimes E \to E$ be on $\mathcal{F} \otimes E \cong E$ is given by ∂ and zero other where and can calling as the trivial representation of L on E. Particularly, the trivial representation of L on $\mathcal{F}$ exists.

Remark (1.41):

If ρ is the representation of L on E and $(\partial, \tilde{f})$ be as in a theorem (1.39).

(i) Thereafter $-\partial^*$ be the differential on E^* and the map $gl(E) \to gl(E^*)$, $g \mapsto -g^*$ be the homeomorphism of the differential graded Lie algebras. Through using $\tilde{f}$ to compose a corresponding weak homeomorphism, which give us the L_∞-algebra homeomorphism $S(L[1]) \to S(gl(E^*)[1])$. We get the produced representation by:

$$\rho^E: S(L[1]) \otimes E^* \to E^*, \qquad a \otimes \eta \mapsto -\rho(a, \cdot)^*\eta, \quad (41)$$

and we defined it as a representation dual to ρ.

(ii) Fixing $n \in \mathbb{Z}$. Then $(-1)^n \downarrow^n \circ\, \partial \circ \uparrow^n$ be the differential on $E[n]$ and $l(E) \to gl(E[n])$, $g \mapsto (-1)^{n|g|} \downarrow^n \circ\, g \circ \uparrow^n$. is the differential graded Lie algebra homomorphism. The produced representation of L on $E[n]$ can be given by:

$$S(L[1]) \otimes E[n] \to E[n], \qquad a \otimes \downarrow^n e \mapsto (-1)^{n+n|a|} \downarrow^n \rho(a, e). \quad (42)$$

3. Simplicial and Cyclic (Co) Homology of *L*-infinity Algebras

Through this section, we study the (co)homology of L_∞-algebra with coefficients in the module and relating the (co)homology of the L_∞-algebra with coefficients in an adjoint representation to the deformation's theory of the L_∞-algebra structure. Then discuss the theory deformations of the L_∞-algebra preserving the invariant inner product. We use references: [7], [10], [12], [19] and [25].

For the algebra we mean the vector space E together with the linear map $\xi: E \otimes E \to E$, the multiplication ξ in general not associative. The graded algebra E is algebra which also represents the graded vector space where in the multiplication is the degree preserving. If $xy = (-1)^{|x||y|}yx \ \forall\, x, y \in E$, we say that E commutative. The homeomorphism of the graded algebras is the homomorphism of degree preserving algebra. The ideal I (for two-sides) in E is called a homogeneous if $I \subset E$ is the graded subspace. The ideal is homogeneous iff it is extended by homogeneous elements. Let (E, m) be an involutive L_∞-algebra in our work.

Corollary (2.1):

For each $p \in \mathbb{Z}$, the symmetric linear maps $\delta: (E\,[1])^{\otimes n} \to E\,[1]$ of degree p and the anti-symmetric linear maps l: $E^{\otimes n} \to E$ of degree $(p + 1 - n)$ have a one-to-one correspondence can given by the formula:

$$l = \uparrow \circ\, \delta\, \circ \downarrow^{\otimes n}, \qquad \delta = (-1)^{\frac{k(k-1)}{2}} \downarrow \circ\, l \circ \uparrow^{\otimes n}. \tag{43}$$

The differential on a graded vector space E be a linear map $d: E \to E$ of degree one such that $d^2 = 0$. And the pair (E, d) is called The *DG* vector space. The homomorphism between the *DG* vector spaces (E, d) and (H, d') is the degree preserving linear map $f: E \to H$ such that: $d' \circ f = f \circ d$. Sometimes, the *DG* vector spaces are called the co-chain complexes.

Given the co-chain complex (E, d), then calls the *n*-cocycle element $e \in E_n$ when $d(e) = 0$ and an n-coboundary when $e = d(h)$ for some $h \in E_{n-1}$. A graded vector space $H(E) = ker(d)/\, im(d)$ checks a non-exactness of the sequence:

$$\dots \xrightarrow{d} E^{n-1} \xrightarrow{d} E^{n} \xrightarrow{d} E^{n+1} \xrightarrow{d} \dots . \tag{44}$$

called a cohomology of (E, d) denoted by: $H^n(E) = \frac{n-cocycles}{n-coboundaries}$ the n<u>th</u> cohomology group.

In the otherwise, the homology of (E, d) measures the non-exactness of the sequence:

$$\dots \xleftarrow{d} E_{n-1} \xleftarrow{d} E_{n} \xleftarrow{d} E_{n+1} \xleftarrow{d} \dots , \tag{45}$$

defined by $H_n(E) = \frac{n-cycles}{n-boundaries}$ the n<u>th</u> homology group.

Definition (2.2):

By recalling Corollary (2.1), the anti-symmetric map l_k: $L^{\otimes k} \to L$ of the degree $2 - k$ is equivalent to the symmetric map of degree one $\delta_k: S^k(L[1]) \to L[1]$ such as:

$$l_k = \uparrow \circ\, \delta_k \circ \downarrow^{\otimes k}. \tag{46}$$

Now, if we write (28) in a part of δ_k, we have the characterization of L_∞-structures on L in parts of a symmetric brackets. For an arbitrary n , then:

$$\sum_{j+i=n+1}\sum_{\Omega\in Sh_{i,n-i}^{-1}}(-1)^{(j-1)i}l_j(l_i\otimes id_L^{\otimes(j-1)})\hat{a}(\Omega)$$
$$=0, \qquad (47)$$

with $\downarrow$ and $\uparrow^{\otimes n}$ are isomorphism, that is corresponding to:

$$\begin{aligned}0&=(-1)^{\frac{n(n-1)}{2}}\sum_{j+i=n+1}\sum_{\Omega\in Sh_{i,n-i}^{-1}}(-1)^{(j-1)i}\downarrow\circ\, l_j\left(l_i\otimes id_L^{\otimes(j-1)}\right)\hat{a}(\Omega)\circ\uparrow^{\otimes n}\\&=(-1)^{\frac{n(n-1)}{2}}\sum_{j+i=n+1}\sum_{\Omega\in Sh_{i,n-i}^{-1}}(-1)^{(j-1)i}\delta_j\circ\downarrow^{\otimes j}\left(l_i\otimes id_L^{\otimes(j-1)}\right)\uparrow^{\otimes n}\circ\,\hat{\varepsilon}(\Omega)\\&=(-1)^{\frac{n(n-1)}{2}}\sum_{j+i=n+1}\sum_{\Omega\in Sh_{i,n-i}^{-1}}\delta_j\big((\downarrow\circ\, l_i)\otimes\downarrow^{\otimes(j-1)}\big)\uparrow^{\otimes n}\circ\,\hat{\varepsilon}(\Omega)\\&=\sum_{j+i=n+1}\sum_{\Omega\in Sh_{i,n-i}^{-1}}\delta_j\left(\delta_i\otimes id_{L[1]}^{\otimes(j-1)}\right)\hat{\varepsilon}(\Omega)\,,\end{aligned}$$

Proposition (2.3):[17]

The L_∞-structure: $\{l_k|1\leq k<\infty\}$ on a graded vector space L is corresponding to the system of linear maps $\delta_k : S^k(L[1])\to L[1]$ for $1\leq k<\infty$, all of degree one, such that:

$$\sum_{i+j=n+1}\sum_{\Omega\in Sh_{i,n-i}^{-1}}\varepsilon(\sigma\Omega)\,,$$
$$\delta_j\big(\delta_i(a_{\Omega(1)},\cdots,a_{\Omega(i)}),a_{\Omega(i+1)},\cdots,a_{\Omega(n)}\big)=0\quad\forall\ n\geq1\,,a_1,\cdots,a_n\in L[1] \qquad (49)$$

Corollary (2.4):

The L_∞-structure on a graded vector space L is corresponding to the linear map of a degree one $\delta:\bar{S}(L[1])\to L[1]$ with the condition: $\delta\circ\mu_S(\delta\otimes id_S)\Delta_S=0$.

Proof:

From Proposition (2.3) we get the element: $\delta=\sum_k\delta_k\in Hom(\bar{S}(L[1]),L[1])$ from degree one.

Through misuse of documents, then we refer to the pair $(L[1],\delta)$ as the L_∞-algebra. Then the L_∞-algebras strict homomorphisms are shown as the degree preserving linear maps such that preserve a symmetric bracket.

Proposition (2.5):[4]

For any two L_∞-algebras $(L[1],\delta)$ and $(L'[1],\delta')$, there exists a one-to-one correspondence among the homeomorphisms of strict L_∞-algebra $f: L\to L'$ and the linear degree preserving maps $g=\downarrow\circ f\circ\uparrow : L[1]\to L'[1]$ satisfy:

$$g\circ\delta_k=\delta'_k\circ g^{\otimes k}\,,\ \forall\ k\geq1, \qquad (50)$$

can be written like:

$$g\circ\delta=\delta'\circ S(g). \qquad (51)$$

Theorem (2.6):

The L_∞-structure on the graded vector space L is corresponding to the codifferential $d \in Coder(S(L[1]))$ together with $d(1) = 0$. In this instance, also can refer to a pair $(S(L[1]), d)$ like an L_∞-algebra.

Corollary (2.7):

When L be of finite kind and of $\mathbb{Z}_{\leq 0}$ -graded, the L_∞-structure on L is corresponding to the differential on a graded $S(L[1]^*)$. Explicatively, let $d_{CE} = -d^*$ for d like in the theorem (2.6).

Definition (2.8):

The L_∞-algebra L is a $\mathbb{Z}_{\leq 0}$-graded finite kind and E to be finite-dimensional and trivial in a negative degree. Then we get the differential graded algebras $S(L[1])^*$ and $(S(L[1]) \otimes E^*)^*$ by the following statement:

$$S(L[1])^* \cong S(L[1]^*)\,, E \cong E^{**}, \text{ and } (S(L[1]) \otimes E^*)^* \cong S(L[1]^*) \otimes E\,.$$

Assume $d_{CE} = -d^*$ define a differential on $S(L[1]^*)$. A map:

$$S(L[1]^*) \otimes (S(L[1]^*) \otimes E) \to S(L[1]^*) \otimes E, \quad (\mu \otimes (\Omega \otimes e)) \mapsto (\mu \vee \Omega) \otimes e, \quad (52)$$

Send the term $S(L[1]^*) \otimes E$ into a left $S(L[1]^*$-module. The linear map $D_{CE}: S(L[1]^*) \otimes E \to S(L[1]^*) \otimes E$ of a degree one the derivation of $S(L[1]^*) \otimes E$ expanding D_{CE} when:

$$D_{CE}\big(\mu \vee (\Omega \otimes e)\big) = D_{CE}\, \mu \vee (\Omega \otimes e) + (-1)^{|\mu|} \mu \vee D_{CE}(\Omega \otimes e), \quad (53)$$

occurs, $\forall\ \mu, \Omega \in S(L[1]^*)\ , e \in E$ homogeneous.

Proposition (2.9):

The representation ρ of the L_∞-algebra L on E is equivalent to the derivation $D_{CE}: S(L[1]^*) \otimes E \to: S(L[1]^*) \otimes E$ expanding d_{CE} with $D_{CE}^2 = 0$. Explicatively, there is $D_{CE} = -D^*$, such that D is a co-derivation expanding d produced by a two-way representation ρ^E.

For a given representation ρ of the L_∞-algebra L on E , then we get $S(L[1]^*) \otimes$ E like our generalized *Chevalley–Eilenberg* complex with D_{CE} as co-boundary operator.

Definition (2.10):

Let E be *Lie* algebra over the field $\mathcal{F}$, with the bracket $[\cdot,\cdot]$. The anti-symmetry of a bracket implies that a bracket is a linear map from $\wedge^2 E \to E$. And make $\mathcal{M}$ be the E module, and also let $C^n(E, \mathcal{M}) = Hom(\wedge^n E, \mathcal{M})$ be a space of anti-symmetric *n*-multi linear functions on E and values within $\mathcal{M}$, what we'll define a module of co-chains of a degree n on E with the values in $\mathcal{M}$. Then a Lie algebra co-boundary operator $d: C^n(E, \mathcal{M}) \to C^{n+1}(E, \mathcal{M})$ is identified by:

$$d\varphi(e_1,\cdots,e_{n+1}) = \sum_{1\le i<j\le n+1} (-1)^{i+j-1}\phi\left([e_i,e_j],e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+1}\right) + \sum_{1\le i\le n+1}(-1)^i e_i\cdot\phi(e_1,\cdots,\hat{e}_i,\cdots,e_{n+1}). \quad (54)$$

Then:

$$H^n(E,\mathcal{M}) = ker(d{:}\,\mathcal{C}^n(E,\mathcal{M}) \to \mathcal{C}^{n+1}(E,\mathcal{M}))/im(d{:}\,\mathcal{C}^{n-1}(E,\mathcal{M}) \to \mathcal{C}^n(E,\mathcal{M})). \quad (55)$$

Be the cohomology of Lie algebra of E with coefficients in $\mathcal{M}$. When $\mathcal{M} = E$, and the action of E on itself is an adjoint action, then we can rewrite $\mathcal{C}^n(E,E)$ simply as $\mathcal{C}^n(E)$, and likewise $H^n(E,E)$ can be denoted by $H^n(E)$.

The relation between the Lie algebra cohomology and E deformations is given by $H^2(E)$. If the bracket in E is denoted by l, and product with infinitesimal deformation by $l_t = l + t\omega$, with $t^2 = 0$, then a map $\omega : \Lambda^2 E \to E$ is a cocycle, and a deformations of the trivial are co-boundaries. By the *Jacobi-identity* we get:

$$l_t\big(e_1, l_t(e_2,e_3)\big) = l_t(l_t(e_1,e_2),e_3) + l_t\big(e_2, l_t(e_1,e_3)\big). \quad (56)$$

Expanding the above expression, we decide that:

$$[e_1,\omega(e_2,e_3)] + \omega(e_1,[e_2,e_3]) = [\omega(e_1,e_2),e_3] + \omega([e_1,e_2],e_3) + [e_2,\omega(e_1,e_3)] + \omega(e_2,[e_1,e_3]). \quad (57)$$

This can be expressed as the condition $d\omega = 0$.

There is a linear bijection $\rho_t{:}\,A \to A$ such that $l_t\big(\rho_t(e_1),\rho_t(e_2)\big) = \rho_t([e_1,e_2])$. And can write $\rho_t = I + t\gamma$, where, $\gamma{:}\,E \to E$ is a linear map. Then:

$$l_t(e_1,e_2) = l(e_1,e_2) + t(d\gamma)(e_1,e_2). \quad (58)$$

Consider the invariant inner product on E, by which we mean the non-degenerate symmetric bi-linear form $\langle .,. \rangle\ E \otimes E \to \mathcal{F}$ which satisfies:

$$\langle [e_1,e_2],e_3\rangle = \langle e_1,[e_2,e_3]\rangle. \quad (59)$$

Then for an invariant inner product, a tensor $\tilde{l}$ is given by:

$$\tilde{l}(e_1,e_2,e_3) = \langle [e_1,e_2],e_3\rangle. \quad (60)$$

is also anti-symmetric, so that $\tilde{l} \in Hom(\Lambda^3 E,\mathcal{F})$. We have that:

$$\tilde{l}(e_1,e_2,e_3) = \tilde{l}(e_3,e_1,e_2). \quad (61)$$

so $\tilde{l}$ is invariant under a cyclic permutation of e_1, e_2, e_3. These deformations are given by $H^3(E,\mathcal{F})$, the co-homology of E with trivial coefficients. To study that connection, we define a map $\omega \in \mathcal{C}^n(E)$ to be a cyclic with respect to,

$$\langle \omega(e_1,\cdots,e_n),e_{n+1}\rangle = (-1)^n\langle e_1,\omega(e_2,\cdots,e_{n+1})\rangle. \quad (62)$$

Then it's easy to see that ω is cyclic if and only if the map $\widetilde{\omega}: E^n \to E$ given by:

$$\widetilde{\omega}(e_1,\cdots,e_{n+1}) = \langle \omega(e_1,\cdots,e_n), e_{n+1}\rangle. \tag{63}$$

is anti-symmetric, by another words, $\widetilde{\omega} \in Hom(\wedge^{n+1} E, \mathcal{F})$. A term cyclic here is used to explains the fact that $\widetilde{\omega}$ be cyclic, it means that:

$$\widetilde{\omega}(e_1,\cdots,e_{n+1}) = (-1)^n \widetilde{\omega}(e_{n+1}, e_1,\cdots,e_n). \tag{64}$$

which holds for any anti-symmetric form.

<u>Note that:</u> [23]

The inner product is not degenerate, then the function $\omega \mapsto \widetilde{\omega}$ is an isomorphism between the sub-space $\mathcal{CC}^n(E)$ consists of cyclic co-chains, and $C^{n+1}(E,\mathcal{F})$. We will see that $d\omega$ is cyclic when ω is cyclic, therefore we could define the cyclic cohomology of the Lie algebra to be:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{CC}^{02}(E) & \xleftarrow{1-t} & \mathcal{CC}^{12}(E) & \xleftarrow{N} & \mathcal{CC}^{22}(E) & \xleftarrow{1-t} & \mathcal{CC}^{32}(E) & \leftarrow & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
\mathcal{CC}^{01}(E) & \xleftarrow{1-t} & \mathcal{CC}^{11}(E) & \xleftarrow{N} & \mathcal{CC}^{21}(E) & \xleftarrow{1-t} & \mathcal{CC}^{31}(E) & \leftarrow & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
\mathcal{CC}^{00}(E) & \xleftarrow{1-t} & \mathcal{CC}^{10}(E) & \xleftarrow{N} & \mathcal{CC}^{20}(E) & \xleftarrow{1-t} & \mathcal{CC}^{30}(E) & \leftarrow & \cdots
\end{array}$$

$$H\mathcal{C}^n(E) = ker(d: \mathcal{CC}^n(E) \to \mathcal{CC}^{n+1}(E))/im(d: \mathcal{CC}^{n-1}(E) \to \mathcal{CC}^n(E)). \tag{65}$$

Also we will display that the isomorphism between $\mathcal{CC}^n(E)$ and $C^{n+1}(E,\mathcal{F})$ commutes with the co-boundary operator, therefore the cohomology of the complex of cyclic co-chains coincides with the Lie algebra cohomology of trivial coefficients. That is, $H\mathcal{C}^n(E) \simeq H^{n+1}(E,\mathcal{F})$. Consequently, away from case of associative algebras, cyclic co-homology does not cause anything new. Seeing these facts, notice that if ω is cyclic, then:

$$\begin{aligned}
&\langle [e_i, \omega(e_1,\cdots,\hat{e}_i,\cdots,e_{n+1})], e_{n+2}\rangle = -\langle [\omega(e_1,\cdots,\hat{e}_i,\cdots,e_{n+1}), e_i], e_{n+2}\rangle \\
&= -\langle \omega(e_1,\cdots,\hat{e}_i,\cdots,e_{n+1}), [e_i, e_{n+2}]\rangle = -\widetilde{\omega}(e_1,\cdots,\hat{e}_i,\cdots,e_{n+1}, [e_i, e_{n+2}]) \\
&\qquad = (-1)^{n+1}\widetilde{\omega}([e_i, e_{n+2}], e_1,\cdots,\hat{e}_i,\cdots,e_{n+1}).
\end{aligned} \tag{66}$$

Thus;

$$d\widetilde{\omega}(e_1,\cdots,e_{n+2}) = \langle d\omega(e_1,\cdots,e_{n+1}),e_{n+2}\rangle$$
$$= \sum_{1\le i<j\le n+1} (-1)^{i+j-1}\langle\omega([e_i,e_j],e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+1}),e_{n+2}\rangle$$
$$+ \sum_{1\le i\le n+1} (-1)^{i}\langle[e_i,\omega(e_1,\cdots,\hat{e}_i,\cdots,e_{n+1})],e_{n+2}\rangle$$
$$= \sum_{1\le i<j\le n+1} (-1)^{i+j-1}\widetilde{\omega}([e_i,e_j],e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+1},e_{n+2})$$
$$+ \sum_{1\le i\le n+1} (-1)^{i+(n+2)-1}\widetilde{\omega}([e_i,e_{n+2}],e_1,\cdots,\hat{e}_i,\cdots,e_{n+1})$$
$$= \sum_{1\le i<j\le n+2} (-1)^{i+j-1}\widetilde{\omega}([e_i,e_j],e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+1},e_{n+2})$$
$$= d\widetilde{\omega}(e_1,\cdots,e_{n+2}). \tag{67}$$

The final equality occurs from the triviality of the operation of E on $\mathcal{F}$, just so the second term in the co-boundary operator description is dropped. Since $d\widetilde{\omega}$ is anti-symmetric, so it follows that $d\omega$ is cyclic.
the deformation $l_t = l + t\omega$ of the Lie algebra is said to keep the inner product if the inner product remains constant for l_t. This happens exactly when ω is cyclic as regards the inner product. Likewise, the trivial deformation which keeps the inner product is given by the linear map $\rho_t = I + t\gamma$ which satisfy: $\langle\rho_t(e_1),\rho_t(e_2)\rangle = \langle e_1,e_2\rangle$.
which is equivalent to the condition $\langle\gamma(e_1),e_2\rangle = -\langle e_1,\gamma(e_2)\rangle$; in another words, γ is cyclic.

The cyclic cohomology $HC^2(E)$ describes the deformations of E which keeps the inner product. Since $HC^2(E) \cong H^3(E,\mathcal{F})$, we see that the cyclic cohomology is independent of the inner product. Specifically, isomorphism depends on the inner product. Therefore, we can describe cyclic cohomology to be the cohomology on $C(E,\mathcal{F})$ produced by the isomorphism between $CC(E)$ and $C(E,\mathcal{F})$. This is a cohomology what we display coincides with $H(E,\mathcal{F})$, so it's independent from the inner product. Then, eventually, we introduce those notations which will promote the generalization of the meaning of the co-boundary operator to the $\mathbb{Z}_2$-graded case. Let $Sh(p,q)$ be the un-shuffles of forms it, the subset of the permutations Ω of $(p+q)$ elements like that $\Omega(i) < \Omega(i+1)$ when $i \ne p$. Let $(-1)^{\Omega}$ is the sign of the permutation. Then,

$$d\omega(e_1,\cdots,e_{n+1})$$
$$= \sum_{\Omega\in Sh(2,n-1)} (-1)^{\Omega}\omega([e_{\Omega(1)},e_{\Omega(2)}],e_{\Omega(3)},\cdots,e_{\Omega(n+1)})$$
$$- (-1)^{n-1} \sum_{\Omega\in Sh(n,1)} (-1)^{\Omega}[\omega(e_{\Omega(1)},\cdots,e_{\Omega(n)}),e_{\Omega(n+1)})]. \tag{68}$$

Let us present the $\mathbb{Z}$-grading on the $\mathcal{F}$-module $C^*(E) = \otimes_{k=1}^{\infty} C^k(E)$ by identifying $deg(\omega) = k-1$ if $\varphi \in C^k(E)$. With this grading, $C^*(E)$ becomes a $\mathbb{Z}$-graded Lie algebra, by a bracket defined by:

$$[\omega,\psi](e_1,\cdots,e_{k+l-1}) = \sum_{\Omega\in Sh(l,k-1)} (-1)^{\Omega}\omega\big(\psi(e_{\Omega(1)},\cdots,e_{\Omega(k)}),e_{\Omega(k+1)},\cdots,e_{\Omega(k+l-1)}\big) - (-1)^{(k-1)(l-1)} \sum_{\Omega\in Sh(k,l-1)} (-1)^{\Omega}\psi\big(\omega(e_{\Omega(1)},\cdots,e_{\Omega(l)}),e_{\Omega(l++1)},\cdots,e_{\Omega(n+1)}\big). \quad (69)$$

for $\omega\in\mathcal{C}^l(E)$ and $\psi\in\mathcal{C}^k(E)$. We shall later state that in a more general context, this bracket satisfies the $\mathbb{Z}$-graded *Jacobi -identity*. Bracket description doesn't rely on the fact that E is a Lie algebra, Except for the case of a Lie algebra, easily we see that $d\omega=[\omega,l]$. The immediate consequence of the *Jacobi-identity* defined if $[l,l]=0$, since $deg(l)=1$, so l is an odd mapping, these facts show that $d^2=0$. A generalization of this result is that given a not necessarily invariant inner product on E, the bracket of cyclic elements is again a cyclic element.

Theorem (2.11):

Assume that E is a $\mathcal{F}$-module, and $\langle\cdot,\cdot\rangle$ be an inner product on E. And supposing that $\omega\in\mathcal{C}^l(E)$ and $\psi\in\mathcal{C}^k(E)$ are cyclic. Then also $[\omega,\psi]$ is cyclic. The definition below includes:

$$\widetilde{[\omega,\psi]}(e_1,\cdots,e_{k+l}) = \sum_{\Omega\in Sh(l,k-1)} (-1)^{\Omega}\widetilde{\omega}\big(\psi(e_{\Omega(1)},\cdots,e_{\Omega(k)}),e_{\Omega(k+1)},\cdots,e_{\Omega(k+l)}\big). \quad (70)$$

Therefore, the inner product induces the structure of the graded Lie algebra in $\mathcal{C}^*(E,\mathcal{F})$, given by $[\widetilde{\omega},\widetilde{\psi}]=\widetilde{[\omega,\psi]}$. Further, $l:\wedge^2E\to E$ be a Lie algebra bracket, and the inner product is invariant with regards to this bracket, then the differential d in $\mathcal{C}^n(E,\mathcal{F})$ is given by $d(\widetilde{\omega})=[\widetilde{\omega},\tilde{l}]$, so the isomorphism between $\mathcal{CC}^*(E)$ and $\mathcal{C}^{*+1}(E,\mathcal{F})$, be an isomorphism of differential $\mathbb{Z}$-graded Lie algebras.

$\mathbb{Z}_2$-Graded Lie Algebras

$\mathbb{Z}_2$-graded $\mathcal{F}$-module which equipped with zero degree bracket $l:E\otimes E\to E$, $(l(a,b)=[a,b])$, and graded anti-commutative $([e_1,e_2]=(-1)^{e_1e_2}[e_2,e_1])$ and satisfy the graded *Jacobi-identity*:

$$\big[e_1,[e_2,e_3]\big] = \big[[e_1,e_2],e_3\big] + (-1)^{e_1e_2}\big[e_2,[e_1,e_3]\big]. \quad (71)$$

Is called a $\mathbb{Z}_2$-graded Lie algebra.

The odd brackets can also be considered, but here we require that the bracket has degree zero, so that $|[x,y]|=|x|+|y|$. We can restrict to the case wherever $\mathcal{F}$ is the field of the characteristic zero, but it is also interesting to allow $\mathcal{F}$ to be a $\mathbb{Z}_2$-graded commutative ring, requiring that $\mathcal{F}_0$ be a field of the characteristic zero. The graded anti-symmetry of the bracket means that the bracket is a linear map $[\cdot,\cdot]\wedge^2E\to E$, where $\wedge^2E$ is the graded wedge product. By calling the graded exterior algebra $\wedge E$ which be defined as a quotient of $\otimes E$ by a graded ideal created by elements of the formula: $x\otimes y+(-1)^{xy}y\otimes x$, for homogeneous elements x,y in E. An element $e_1\wedge\cdots\wedge e_n$ is said to have external degree n and parity internal degree; $|e_1|+\cdots+|e_n|$. If φ has degree $deg(\varphi)$ and parity $|\varphi|$, and similarly for η, then:

$$\varphi\wedge\eta = (-1)^{|\varphi||\eta|+\deg(\varphi)\deg(\eta)}\eta\wedge\varphi. \quad (72)$$

If Ω is a permutation, then can define $\varrho(\Omega; e_1, \cdots, e_n)$ by requiring:

$$e_1 \wedge \cdots \wedge e_n = (-1)^{\Omega} \varrho(\Omega; e_1, \cdots, e_n) e_{\Omega(1)} \wedge \cdots \wedge e_{\Omega(n)}. \tag{73}$$

where $(-1)^{\Omega}$ be the sign of the permutation Ω.

So as see how the co-boundary operator must be updated in the case of $\mathbb{Z}_2$-graded algebras, we consider infinitesimal deformations of the Lie algebra. If we define the deformed bracket by $l_t = l + t\omega$ as before, then we wish the deformed bracket to remain even, so that if t is taken to be an even parameter, then ω must be even. However, if we let t be an odd parameter, then ω must be odd. We also must consider that parameters should graded commute with elements of E, so that $et = (-1)^{ta} te$. The graded *Jacobi-identity* takes the form:

$$l_t(a, l_t(b, c)) = l_t(l_t(a, b), c) + (-1)^{ab} l_t(b, l_t(a, c)). \tag{74}$$

Expanding the formula, we obtain the condition:

$$\omega(l(a,b),c) + (-1)^{bc+1}\omega(l(a,c),b) + (-1)^{a(b+c)}\omega(l(b,c),a) + l(\omega(a,b),c) + (-1)^{bc+1} l(\omega(a,c),b) + (-1)^{a(b+c)} l(\omega(b,c),a) = 0. \tag{75}$$

which can be explained as $d\omega = 0$, if we define for $\omega: \wedge^n E \to E$,

$$d\omega(e_1, \cdots, e_{n+1}) = \sum_{\Omega \in Sh(2,n-1)} (-1)^{\Omega} \varrho(\Omega) \omega([e_{\Omega(1)}, e_{\Omega(2)}], e_{\Omega(3)}, \cdots, e_{\Omega(n+1)}) - (-1)^{n-1} \sum_{\Omega \in Sh(n,1)} (-1)^{\Omega} \varrho(\Omega) [\omega(e_{\Omega(1)}, \cdots, e_{\Omega(n)}), e_{\Omega(n+1)}]. \tag{76}$$

In general, when $\mathcal{M}$ be a $\mathbb{Z}_2$-graded E module, we can describe the right multiplication $\mathcal{M} \otimes E \to \mathcal{M}$ by $m \cdot a = -(-1)^{ma} a \cdot m$. Then if we let $\mathcal{C}^n(E, \mathcal{M}) = Hom(\wedge^n E, \mathcal{M})$ be the co-chains module of degree n on E with the values in $\mathcal{M}$, we can define a coboundary operator $d : C^n(E, \mathcal{M}) \to C^{n+1}(E, \mathcal{M})$ by:

$$\begin{aligned} d\omega(e_1, \cdots, e_{n+1}) &= \sum_{\Omega \in Sh(2,n-1)} (-1)^{\Omega} \varrho(\Omega) \omega([e_{\Omega(1)}, e_{\Omega(2)}], e_{\Omega(3)}, \cdots, e_{\Omega(n+1)}) \\ &\quad - (-1)^{n-1} \sum_{\Omega \in Sh(n,1)} (-1)^{\Omega} \omega(e_{\Omega(1)}, \cdots, e_{\Omega(n)}) \\ &\quad \cdot e_{\Omega(n+1)}. \end{aligned} \tag{77}$$

Then we define $H^n(E, \mathcal{M})$ in the same manner as for ordinary Lie algebras, and as before, denote $\mathcal{C}^n(E, E) = \mathcal{C}^n(E)$ and $H^n(E) = H(E, E)$ for the adjoint action of E.

As the case of Hochschild Lie algebras, we obtain that trivial deformations are given by means of co-boundaries of linear maps $\gamma: E \to E$. Thus, the deformations of infinite small of a $\mathbb{Z}_2$-graded Lie algebra which be classified by $H^2(E)$, where the Lie algebra co-homology is given by means of the co-boundary operator in above equation. Factually $d^2 = 0$ is verified using the normal method.

$\mathcal{C}^*(E) = \oplus_{K=1}^{\infty} C^*(E)$, has a natural $\mathbb{Z}_2 \times \mathbb{Z}$ grading, with the bidegree of ahomogeneous element $\omega \in \mathcal{C}^n(E)$ given by $bid(\omega) = (|\omega|, n-1)$. The operation of the natural bracket in $\mathcal{C}^*(E)$, is given by:

$$[\omega, \psi](e_1, \cdots, e_{k+l-1}) =$$

$$\sum_{\Omega\in Sh(l,k-1)} (-1)^{\Omega}\varrho(\Omega)\omega\big(\psi\big(e_{\Omega(1)},\cdots,e_{\Omega(k)}\big),e_{\Omega(k+1)},\cdots,e_{\Omega(k+l-1)}\big)$$

$$-(-1)^{\varphi\psi+(k-1)(l-1)}\sum_{\Omega\in Sh(k,l-1)} (-1)^{\Omega}\varrho(\Omega)\psi\big(\omega\big(e_{\Omega(1)},\cdots,e_{\Omega(l)}\big),e_{\Omega(l+1)},\cdots,e_{\Omega(n+1)}\big). \tag{78}$$

The bracket makes $C^*(E)$ into the $\mathbb{Z}_2\times\mathbb{Z}$-graded Lie algebra. The differential has the form $d(\omega)=[\omega,l]$, and the condition $[l,l]=0$ is exactly equivalent to the $\mathbb{Z}_2$ −graded *Jacoby-identity* for l. Since l even has parity, we notice this as a $\mathbb{Z}_2\times\mathbb{Z}$-graded map, it is odd, since $\langle(0,1),(0,1)\rangle=1$.

For a $\mathbb{Z}_2$-graded $\mathcal{F}$-module E, an inner product is a (right) $\mathcal{F}$-module homomorphism $h: E\otimes E\to E$, which is symmetric, and non-degenerate. The inner product is denoted by $\langle e,v\rangle = h(e\otimes v)$. Graded symmetry means that $\langle e,v\rangle=(-1)^{ev}\langle v,e\rangle$. Non-degeneracy means that the map $\gamma: E\to E^*=Hom(E,\mathcal{F})$, given by $\gamma(e)(v)=\langle e,v\rangle$ is an isomorphism. If h is an even map, then the inner product is even. We will consider only even inner products on E. If E be the $\mathbb{Z}_2$-graded Lie algebra, then the notion of the invariant inner product as in a non-graded case can be defined by $\langle[e_1,e_2],e_3\rangle=\langle e_1,[e_2,e_3]\rangle$. Then the tensor $\tilde{l}$, given by:

$$\tilde{l}(e_1,e_2,e_3)=\langle[e_1,e_2],e_3\rangle. \tag{79}$$

is antisymmetric, so that $\tilde{l}\in Hom(\wedge^3 E,\mathcal{F})$. We got:

$$\tilde{l}(e_1,e_2,e_3)=(-1)^{e_3(e_1+e_2)}\tilde{l}(e_3,e_1,e_2). \tag{80}$$

so $\tilde{l}$ satisfy a graded invariance property under cyclic permutations. Generally, we say that an element $\omega\in C^n(E)=Hom(\wedge^n E,E)$ is cyclic with regards to the inner product, when:

$$\langle\omega(e_1,\cdots,e_n),e_{n+1}\rangle=(-1)^{n+e_1\omega}\langle e_1,\omega(e_2,\cdots,e_{n+1})\rangle. \tag{81}$$

Then ω is cyclic iff $\widetilde{\omega}:\wedge^{n+1}E\to E$, presented by:

$$\widetilde{\omega}(e_1,\cdots,e_{n+1})=\langle\omega(e_1,\cdots,e_n),e_{n+1}\rangle. \tag{82}$$

is anti-symmetric. As the inner product is not degenerate, the map $\omega\mapsto\widetilde{\omega}$ is an even isomorphism between the sub-module $CC^n(E)$ of $C^n(E)$ consists of cyclic elements, and $C^{n+1}(E,\mathcal{F})$.

Theorem (2.12):

Assuming that E is a $\mathbb{Z}_2$-graded on $\mathcal{F}$-module, and $\langle\cdot,\cdot\rangle$ is the inner product on E. And supposing that $\omega\in C^l(E)$ and $\psi\in C^k(E)$ be cyclic. Then also $[\omega,\psi]$ be cyclic. Moreover, the following formula includes:

$$\widetilde{[\omega,\psi]}(e_1,\cdots,e_{k+l})=\sum_{\Omega\in Sh(l,k-1)}(-1)^{\Omega}\varrho(\Omega)\widetilde{\omega}\big(\psi\big(e_{\Omega(1)},\cdots,e_{\Omega(k)}\big),e_{\Omega(k+1)},\cdots,e_{\Omega(k+l)}\big). \tag{83}$$

As a consequence, the inner product produces a structure of the $\mathbb{Z}_2\times\mathbb{Z}$-graded Lie algebra in $C^*(E,\mathcal{F})$, given by $[\widetilde{\omega},\widetilde{\psi}]=\widetilde{[\omega,\psi]}$. Further, $l:\wedge^2 E\to E$ is the $\mathbb{Z}_2$-graded Lie algebra bracket, and the inner product is invariant with regard to this bracket, then the differential d in $C^n(E,\mathcal{F})$is given by $d(\widetilde{\omega}) = [\widetilde{\omega}, \tilde{l}]$, so the isomorphism between $\mathcal{CC}^*(E)$ and $\mathcal{C}^{*+1}(E,\mathcal{F})$, is the isomorphism of differential $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie algebras.

4. Dihedral Homology of *L*-infinity Algebra

In this section, we discuss and study the dihedral cohomology of $L - infinity$ algebras. So, we define and study some properties of the cyclic and dihedral cohomology theory of $L - infinity$ algebras. We use the references: [1], [2], [3], [14], [22] and [30].

Proposition (3.1):

If (E, m) is L_∞-algebra with unity. Then the graded vector space $\mathcal{CC}^\bullet(E)$ called the cyclic group of $\mathcal{C}_i$ order i and defined by the formula:

$$\mathcal{CC}^\bullet(E) = \sum \prod_{i=0}^{\infty} [(\Sigma^{-1}E^*)^{\otimes i}]_{\mathbb{C}_i} \tag{84}$$

Then a derivation m on $\hat{T}\Sigma^{-1}E^*$ caused a well–defined derivation of quotient $\mathcal{CC}^\bullet(E)$.

Proof:

Let W be a sub-space of $\hat{T}\Sigma^{-1}E^*$ of approximate sums of elements of the formula: $ab - (-1)^{|a||b|}ba$. Then $\mathcal{CC}^\bullet(E) = \hat{T}\Sigma^{-1}E^*/m$.

Definition (3.2):

Let E be the graded vector space with an involution. The dihedral group by order $2n$ denoted by D_n, given by:

$$D_n = \langle r, s | r^n = s^2 = 1\,, srs^{-1} = r^{-1} \rangle\ . \tag{85}$$

And there exists the following actions of D_n on $E^{\otimes n}$.

(1) Defines the dihedral action by:

$$\begin{aligned} r(e_1 \otimes e_2 \otimes \cdots \otimes e_n) &= (-1)^{\varepsilon} e_n \otimes e_1 \otimes \cdots \otimes e_{n-1}, \\ s(e_1 \otimes e_2 \otimes \cdots \otimes e_n) &= (e_1 \otimes e_2 \otimes \cdots \otimes e_n)^*. \end{aligned} \tag{86}$$

(2) Defines the skew-dihedral action by:

$$\begin{aligned} r\,(e_1 \otimes e_2 \otimes \cdots \otimes e_n) &= (-1)^{\varepsilon} e_n \otimes e_1 \otimes \cdots \otimes e_{n-1}, \\ s(e_1 \otimes e_2 \otimes \cdots \otimes e_n) &= -\,(e_1 \otimes e_2 \otimes \cdots \otimes e_n)^*. \end{aligned} \tag{87}$$

Then, $\varepsilon = |e_n| \sum_{i=1}^{n-1} |e_i|$ arises as ordinary with the permutation of $e_i \in E$.

Proposition (3.3):

If (E, m) be the involutive L_∞-algebra.

By ${}^{+}\mathcal{CD}^\bullet(E)$ the graded vector space can denote as:

$${}^{+}\mathcal{CD}^\bullet(E) = \sum \prod_{I=1}^{\infty} [(\Sigma^{-1}E^*)^{\otimes i}]_{D_i}. \tag{88}$$

where, D_i be a dihedral group of the order $2i$ taking action of the dihedral action (86). Then the derivation m on $\hat{T}\sum^{-1}E^*$ caused a well–defined derivation of quotient ${}^{+}CD^{\bullet}(E)$.

By ${}^{-}CD^{\bullet}(E)$ the graded vector space can denote as:

$$ {}^{-}CD^{\bullet}(E) = \sum \prod_{i=1}^{\infty} [(\Sigma^{-1}E^*)^{\otimes i}]_{D_i}. \tag{89}$$

where D_i be a dihedral group of the order $2i$ taking action of the skew-dihedral action (87). Then the derivation m on $\hat{T}\sum^{-1}E^*$ caused a well–defined derivation of quotient ${}^{+}CD^{\bullet}(E)$.

Proof:

Let $\mathcal{M}$ be a sub-space of $\hat{T}\sum^{-1}E^*$ of approximate sums of elements of forms: $ab - (-1)^{|a||b|}ba$ and $a - a^*$. Then the dihedral category of E is defined in the formula ${}^{+}CD^{\bullet}(E) = \sum \hat{T}\sum^{-1}E^* / \mathcal{M}$. Since m is involutive $(a - a^*) = m(a) - m(a)^* \in \mathcal{M}$, this means $m(\mathcal{M}) \subset \mathcal{M}$ and then there's the first part. The second part essentially is the same.

Definition (3.4):

For an involutive L_∞-algebra (E, m) we have the following statements:

1. The complex of dihedral cohomology of E is a DG vector space ${}^{+}CD^{\bullet}(E)$ with differential produced by m like in a proposition (3.3). A cohomology of this complex is indicated by ${}^{+}HD^{\bullet}(E)$.
2. The complex of skew-dihedral cohomology of E is a DG vector space ${}^{-}CD^{\bullet}(E)$ with differential produced by m like in a proposition (3.3). A cohomology of this complex is indicated by ${}^{-}HD^{\bullet}(E)$.

Theorem (3.5):

For an involutive L_∞-algebra (E, m) the Hochschild cohomology of E introduce in the decompose:

$$HH^{\bullet}(E) \cong {}^{+}HH^{\bullet}(E) \otimes {}^{-}HH^{\bullet}(E). \tag{90}$$

Theorem (3.6):

For an involutive L_∞-algebra (E, m) the cyclic cohomology of E can be written as form:

$$HC^{\bullet}(E) \cong {}^{+}HD^{\bullet}(E) \otimes {}^{-}HD^{\bullet}(E). \tag{91}$$

Proof:

Since $D_n = \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, the complexes ${}^{+}CD^{\bullet}(E)$ and ${}^{-}CD^{\bullet}(E)$ are a quotients of $CC^{\bullet}(E)$ by the two various actions of $\mathbb{Z}_2$ originating from an involution on E. By a co-invariants isomorphism with invariants those spaces are defined by the *Eigen* spaces of an eigenvalues $+1$ and -1 of an involution on $CC^{\bullet}(E)$ and as for theorem (3.5), the results precede.

Particularly, the cyclic cohomology of a $\mathbb{Z}_2$-graded Lie algebra can define in the same way as for an ordinary Lie algebra. The isomorphism: $HC^n(E) \cong H^{n+1}(E, \mathcal{F})$ holds for the $\mathbb{Z}_2$-graded case as well.

Theorem (3.7):

Let e be lie algebra with unity and involution with module $\mathcal{M}$, then there is a relation between a cyclic cohomology $HC(e, \mathcal{M})$ and a dihedral cohomology $HD(e, \mathcal{M})$ is defined as sequence:

$$\cdots \to {}^{-}HD^{n+1}(e,\mathcal{M}) \longrightarrow {}^{+}HD^{n}(e,\mathcal{M}) \xrightarrow{s} HC^{n}(e,\mathcal{M}) \xrightarrow{t} {}^{-}HD^{n}(e,\mathcal{M}) \to \cdots.$$

Proof:

We can get the required by getting the long exact sequence:

$$\cdots \to {}^{-}HD^{n+1}(e,\mathcal{M}) \longrightarrow {}^{+}HD^{n}(e,\mathcal{M}) \xrightarrow{s} HC^{n}(e,\mathcal{M}) \xrightarrow{t} {}^{-}HD^{n}(e,\mathcal{M}) \to \cdots.$$

From a short exact sequence:

$$0 \longrightarrow Tot^{-}D(e,\mathcal{M}) \to Tot^{+}D(e,\mathcal{M}) \to TotC(e,\mathcal{M}) \xrightarrow{t} 0.$$

Summary and Concluding Remarks

In our paper we are introducing and study the cyclic and dihedral cohomology of L-infinity algebras and we generalize some relations between them. We prove Connes-Tsygan long exact sequence in the dihedral homology of L-infinityalgebras. We introduce some relations in the simplicial, cyclic and dihedral cohomology of L-infinity algebras.

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